A Poisson bracket in multisymplectic field theory

Quantum Theory Seminar

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Plan of this talk

- Basic objects of multisymplectic geometry
- Poisson forms and the Poisson bracket
- Classification of Poisson forms and the scaling degree
- Conclusions

Multisymplectic Formulation of Classical Field Theory

- Main idea: treat translations in all space-time directions x^μ on equal footing
- Obtain a finite-dimensional (multisymplectic) phase space for Field Theory
- Alternative form of field equations

Lagrangean	$L(x^\mu, \varphi^i, \partial_\mu \varphi^i)$
Euler-Lagrange equations	$\frac{\partial L}{\partial \varphi^i} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \varphi^i}$
multimomenta	$\pi_i^\mu = \frac{\partial L}{\partial \partial_\mu \varphi^i}$
De Donder-Weyl Hamiltonian	$H(x^\mu, \varphi^i, \pi_i^\mu) = \pi_i^\mu \partial_\mu \varphi^i - L$
D-W equations	$\frac{\partial H}{\partial \varphi^i} = -\partial_\mu \pi_i^\mu, \qquad \frac{\partial H}{\partial \pi_i^\mu} = \partial_\mu \varphi^i$

- Aim: geometric interpretation

Multiphase space

Mechanics	Field theory
Extended configuration space $Q \times \mathbb{R}$	Bundle E over space-time M^n with fiber Q $E \longrightarrow M$
Double extended phase space $P = T^*Q \times \mathbb{R}^2$	Multiphase space $P = J_1^* E$
Coordinates $t,\, q^i,\, p_i,\, E$	Coordinates $x^\mu,\, q^i,\, p_i^\mu,\, p, \qquad \mu = 1, \ldots, n$
Poincaré-Cartan 1-form $\theta = p_i\, dq^i - E\, dt$	Poincaré-Cartan n-form $\theta = p_i^\mu\, dq^i \wedge d^n x_\mu - p\, d^n x$
Symplectic form $\omega = -d\theta = dq^i \wedge dp_i + dE \wedge dt$ closed, non-degenerate	Multisymplectic form $\omega = -d\theta = dq^i \wedge dp_i^\mu \wedge d^n x_\mu + dp \wedge d^n x$ closed, non-degenerate on vector fields

The jet bundle and its dual

- Local sections φ of $E \stackrel{\pi}{\longrightarrow} M$ define a map

$$\bar{q}:\ T_xM \longrightarrow T_qE\,, \quad \xi \mapsto D\varphi(\xi), \quad q=\varphi(x),$$

  with the property $T\pi \circ \bar{q} = id_{T_xM}$.

- The (first) jet bundle is the set of all such maps. Fibers J^1_qE over E.
  J^1_qE is an affine space, modelled over the vector space $T^*_xM \otimes V_qE$ of dimension $n \times N$.

$$\bar{q} = \partial_\mu \otimes dx^\mu + q^i_\mu \partial_{q^i} \otimes dx^\mu,$$

- The dual jet bundle consist of al affine maps $J^1_qE \longrightarrow \Lambda^n T^*_xM$.

$$\mathbf{p}:\ \bar{q} \mapsto (p + p^\mu_i q^i_\mu)\, d^n x$$

- Note that $\bar{q}^* dx^\mu = dx^\mu$, $\bar{q}^* dq^i = q^i_\mu dx^\mu$. Hence,

$$\bar{q}^*(p^\mu_i dq^i \wedge d^n x_\mu + p\, d^n x) = (p + p^\mu_i q^i_\mu)\, d^n x = \mathbf{p}(\bar{q}).$$

- Tautological construction gives Poincaré-Cartan and multisymplectic forms

$$\theta_{\mathbf{p}} = \mathbf{p} = p^\mu_i dq^i \wedge d^n x_\mu + p\, d^n x, \qquad \omega = -d\theta.$$

Hamiltonian multivector fields

- We study the equation

$$i_X \omega = df$$

$X \ldots$ Hamiltonian multivector field associated with $f \ldots$ Hamiltonian form.
r-vector field $\Leftrightarrow$ $(n-r)$-form.

- Neither X nor f is uniquely defined. Not every f is a Hamiltonian form.

- Necessary condition for X: $di_X \omega = 0$.
$X \ldots$ locally Hamiltonian multivector field.

- Lie derivative along multivector field $L_X = di_X - (-1)^{|X|} i_X d$.
Locally Hamiltonian multivector fields satisfy $L_X = 0$.

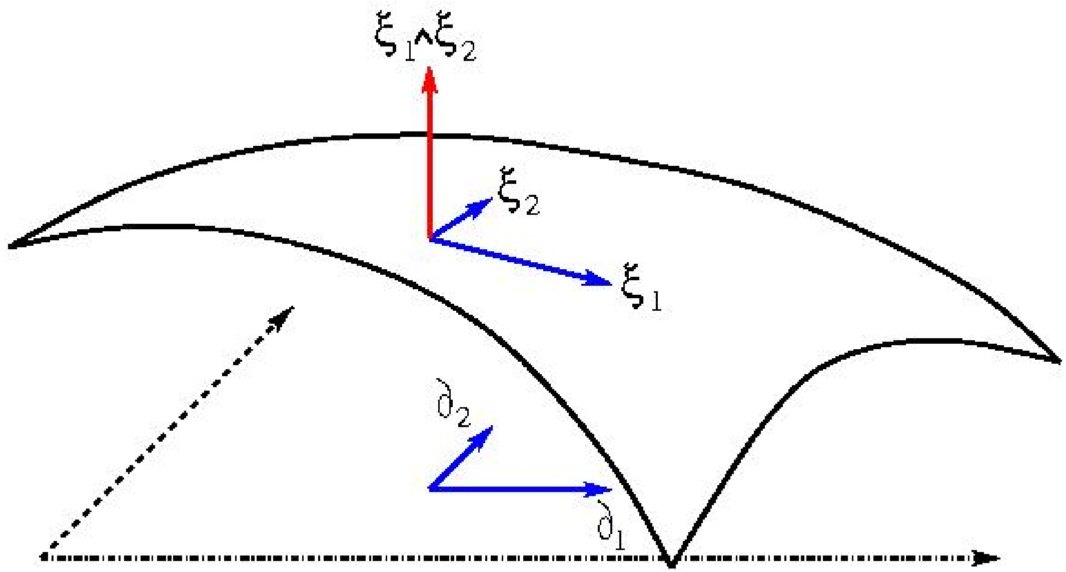
- Sufficient condition for X: $L_X \theta = 0$.
$X \ldots$ exact Hamiltonian multivector field. $f = \pm i_X \theta$.

The universal multimomentum map

- Hamiltonian vector fields are associated with Hamiltonian $(n-1)$-forms. Let ξ be a vector field on E, projectable onto M, i.e. an infinitesimal symmetry on E.
  Theorem [Gotay et al.] Iff ξ is a symmetry of the Lagrangean, $L_\xi L = 0$, then ξ is exact Hamiltonian, $L_\xi \theta = 0$.
  $J(\xi) = i_X \theta$ defines the Noether current:
  $$d(\varphi, \pi)^* J(\xi) = 0.$$
  $(n-1)$-forms can be paired with hyper surfaces in M.
- Generalization: Every exact Hamiltonian multivector field is projectable onto E and M.
  If ξ_1, ξ_2 are exact Hamiltonian vector fields and $[\xi_1, \xi_2] = 0$, then $\xi_1 \wedge \xi_2$ is an exact Hamiltonian bivector field.
- Universal multimomentum map
  $$J(X) = \pm i_X \theta.$$

Hamiltonian n-vector fields

- Hamiltonian functions are associated with Hamiltonian n-vector fields.
  An n-vector field is separable $\Leftrightarrow X = \xi_1 \wedge \cdots \wedge \xi_n \Leftrightarrow n$-distribution on P.
- **Theorem [CP, H. Römer]** The De Donder-Weyl Hamiltonian defines separable Hamiltonian n-vector fields X_h.
  The integral submanifolds of X_h correspond to solutions of the D-W equations for H.



Multivector calculus

- Multivector fields on M are generated by vector fields on M via $\wedge$:

$$X = \sum X^{i_1 \cdots i_r} \xi_{i_1} \wedge \cdots \wedge \xi_{i_r}$$

- Schouten bracket

$$[\cdot,\cdot] \ : \ \Lambda^r \mathfrak{X}(M) \times \Lambda^s \mathfrak{X}(M) \ \longrightarrow \ \Lambda^{r+s-1} \mathfrak{X}(M)$$

  Commutator bracket in vector fields, (graded) Leibniz rule, graded antisymmetry

- Jacobi identity

$$(-1)^{(r-1)(t-1)}[X,[Y,Z]] + \text{cycl. perm.} = 0$$

- Lie derivative along multivector fields

$$L_X = d i_X \pm i_X d$$

  satisfies, e.g.

$$L_{[X,Y]} = \pm L_X L_Y - L_Y L_X$$

The Poisson bracket

- **Definition:** The Poisson bracket for f and g is defined to be

$$\begin{aligned}\{f,g\} &= -L_X g \pm L_Y f \pm L_{X\wedge Y}\theta \\ &= \pm i_X i_Y \omega + d\left(-i_X g \pm i_Y f \pm i_X i_Y \theta\right),\end{aligned}$$

  where $df = i_X\omega$, $dg = i_Y\omega$.

- Well defined for and closes on Poisson forms

$$\text{f is Poisson:} \qquad i_Z\omega = 0 \quad \Rightarrow \quad i_Z f = 0.$$

- The Hamiltonian multivector field for $\{f,g\}$ is $[Y,X]$.

- **Theorem [Forger, CP, Römer]:** The Poisson bracket is graded antisymmetric and satisfies a graded Jacobi identity

$$\pm\{f,\{g,h\}\} + \text{ cycl. perm.} = 0$$

- θ is necessary!

- The multimomentum map is equivariant: $\{J(X),J(Y)\} = J([Y,X])$.

Classification of Hamiltonian multivector fields

- New ingredient: Scaling vector field

$$\Sigma = p_i^\mu \frac{\partial}{\partial p_i^\mu} + p \frac{\partial}{\partial p}$$

- Σ counts polynomial degree in multimomentum and energy variables, the scaling degree:

$$L_\Sigma(p^k) = k\,p^k, \qquad L_\Sigma(dp_i^\mu) = dp_i^\mu$$

$$L_\Sigma(dq^i) = 0, \qquad \left[\Sigma, \frac{\partial}{\partial p_i^\mu}\right] = -\frac{\partial}{\partial p_i^\mu}.$$

- We have $L_\Sigma \theta = \theta$.

- **Theorem [Forger, CP, Römer]:** Let X be a locally Hamiltonian r-vector field. Then

$$X = X_{-1} + X_0 + \cdots + X_{r-1} + \ker\omega, \qquad [\Sigma, X_\lambda] = \lambda X_\lambda$$

Poisson vs. Hamiltonian forms

- Multivector calculus gives

$$L_\Sigma df = L_\Sigma i_X \omega = i_{[\Sigma,X]+X}\omega$$

Hence

Corollary: Every Hamiltonian $(n-r)$-form can be written as

$$f = f_0 + f_1 + \cdots + f_r + \text{closed}$$

- If $\lambda \geq 0$, then X_λ is (globally) Hamiltonian with associated Hamiltonian form

$$J(X_\lambda) = \pm \frac{1}{\lambda+1} i_{X_\lambda}\theta.$$

- $\Rightarrow$ Every Hamiltonian form of positive scaling degree is equivalent to a Poisson form.

Zero scaling degree: cohomology of configuration bundle E

Conclusions and outlook

- Generalization of symplectic geometry is viable
- Rich structure: Multivector fields, differential forms, Poisson bracket
- What is the role of the scaling degree?
- Coordinate invariant statements, but proofs in coordinates
- Higher order formalism: Einstein-Hilbert Lagrangean
- Product structure?
- Covariant treatment of constraints?
- Quantization à la Cattaneo-Felder?

The multimomentum dependence of locally Hamiltonian multivector fields

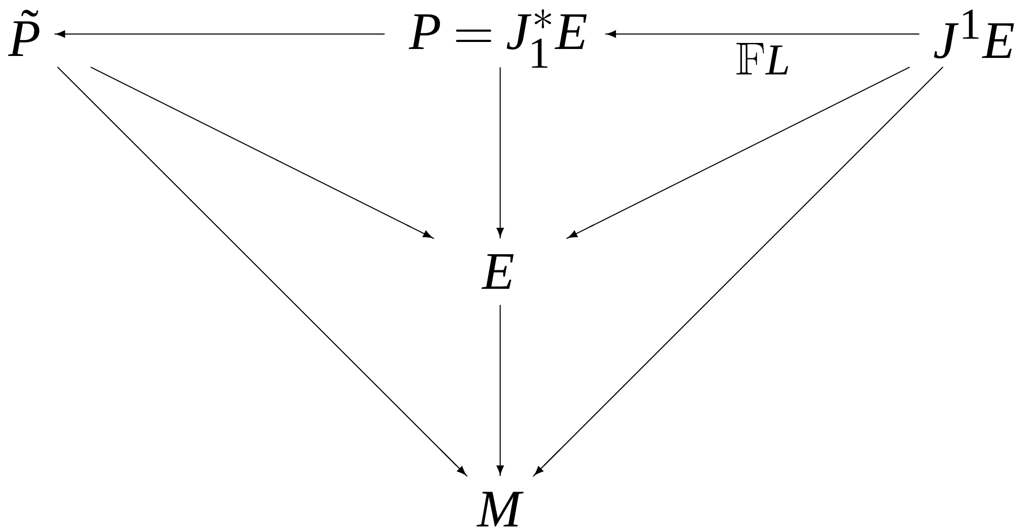
- Coordinate expression for X

$$X = \ldots + \frac{1}{(r-1)!} X^{i,\mu_2\cdots\mu_r} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial x^{\mu_2}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\mu_r}} + \ldots$$

- Antisymmetric polynomial in the multimomentum variables

$$\frac{\partial^2 X^{i,\mu_2\cdots\mu_r}}{(\partial p_j^\nu)^2} = 0, \qquad \text{and} \qquad \frac{\partial X^{i,\mu_2\cdots\mu_r}}{\partial p_j^\nu} = 0 \quad \text{if } \nu \notin \{\mu_2, \cdots, \mu_r\}$$

The bundle structure



M ...space-time,

E ...configuration bundle,

$J^1 E$ ...first jet bundle,

P ...multiphase space,

$\tilde{P}$ ...reduced multiphase space

$\mathbb{F}L$ ...covariant Legendre transformation