Products for the multisymplectic Poisson bracket

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Plan of this talk

- Basic objects of multisymplectic geometry
- Poisson forms and the Poisson bracket
- Classification of Poisson forms
- Products for the Poisson bracket
- Conclusions

Basic objects of multisymplectic geometry

- A multisymplectic manifold is a manifold $\mathcal{P}$ together with a closed, nondegenerate form ω .
- Multisymplectic Field Theory is a special case:

  Multiphase space [Kijowski '73]: $\mathcal{P}\ (x^\mu, q^i, p^\nu_j, p) \longrightarrow \mathcal{E}\ (x^\mu, q^i) \longrightarrow \mathcal{M}^n\ (x^\mu)$

  $$\text{multimomenta } p^\nu_j = \frac{\partial L}{\partial \partial_\nu \varphi^j}, \qquad \text{energy variable } p = \partial_\nu \varphi^j \frac{\partial L}{\partial \partial_\nu \varphi^j} - L$$

  $$\omega = dq^i \wedge dp^\mu_i \wedge d^n x_\mu - dp \wedge d^n x, \qquad \omega = -d\theta$$

  The scaling vector field $\Sigma = p^\mu_i \frac{\partial}{\partial p^\mu_i} + p \frac{\partial}{\partial p}$ satisfies $i(\Sigma)\omega = -\theta$.
- $i(X)\omega = df$ associates **Hamiltonian r-vector fields** X with **Hamiltonian $(n-r)$-forms** f. A locally Hamiltonian multi-vector field X satisfies $di_X\omega = 0$. Neither f nor X are uniquely defined. Not every form f is Hamiltonian.

Poisson forms and the Poisson bracket

- For multi-vector fields X, Y one can define an extension of the Lie bracket of vector fields, the Schouten bracket. It satisfies a graded Jacobi identity

$$\pm[X,[Y,Z]] + \text{cycl. perm} = 0.$$

- Corresponding bracket for Hamiltonian forms?
$\{f,g\} = \pm i(X)i(Y)\omega$ does not satisfy a Jacobi identity.

- Correction terms [Forger, C.P., Römer '01]

$$\{f,g\} = \pm i(X)i(Y)\omega + d(\pm i(Y)f \pm i(X)g \pm i(X)i(Y)\theta)$$

establish Jacobi identity, need to restrict f, g to **Poisson forms**.

$$f \text{ Poisson } \Leftrightarrow f = i(\xi)\omega$$

Then, $i(Y)f$ does not depend on the choice of Y.

Classification of Poisson forms: The scaling degree

Theorem [Forger, C.P., Römer '04]

1. Let X be a locally Hamiltonian r-vector field.

   Then

   $$X = X_{-1} + X_0 + X_1 + \ldots + X_{r-1} + \ker \omega, \text{ where } [\Sigma, X_\lambda] = \lambda X_\lambda.$$

   λ is the scaling degree of X_λ denoted by $\mathfrak{p}(X_\lambda)$.

   Note that λ is bounded by the form degree of X_λ!

2. Decomposition property extends to Poisson forms by $L_\Sigma i_X \omega = i_{[\Sigma,X]+X}\omega$.

3. If $\lambda \geq 0$ then $J(X_\lambda) = \pm\frac{1}{\lambda+1} i(X_\lambda)\theta$ is a Poisson form with associated X_λ

   $$L_\Sigma J(X_\lambda) = (\lambda+1) J(X_\lambda) =: \mathfrak{p}(J(X_\lambda))\, J(X_\lambda)$$

4. $\{J(X_\lambda), J(Y_\mu)\} = J([Y_\mu, X_\lambda]) + \text{ closed term}$. (No extra term for $\lambda = 1$ or $\mu = 1$.)

Products: Degree analysis

- Poisson bracket $\{,\}$ is of tensor degree $-(n-1)$:

$$\{r-\mathrm{form}, s-\mathrm{form}\} = \big(r+s-(n-1)\big)-\mathrm{form}$$

- Poisson bracket $\{,\}$ is of scaling degree -1:

$$\mathfrak{p}(\{f,g\}) = \mathfrak{p}(f)+\mathfrak{p}(g)-1$$

- If a product adds the scaling degree, it has to be of tensor degree $-n$ (or lower):

$$n-r+n-s \geq \mathfrak{p}(f)+\mathfrak{p}(g) = \mathfrak{p}(f \bullet g) \leq n-(r+s+?)$$

A new product

- Natural candidate is $X \wedge Y$:

$$[X \wedge Y, Z] = \pm [Z, X] \wedge Y + X \wedge [Y, Z]$$

- Problem: For X, Y Hamiltonian, $X \wedge Y$ is not necessarily locally Hamiltonian. Sufficient: $[X, Y] = 0$.

- Solution:

$$X_\lambda \bullet Y_\mu = X_\lambda \wedge Y_\mu - \frac{(-1)^s}{\lambda+\mu+1} [X, Y] \wedge \Sigma$$

  is locally Hamiltonian.

- But: $J(X) \bullet J(Y) = J(X \bullet Y)$ does not satisfy Leibniz rule (because the $J(X)$ do not close). Modification of Poisson bracket?

- The Hamiltonian form for $X_\lambda \bullet X_\mu$ is of scaling degree $\lambda + \mu + 1 = (\lambda + 1) + (\mu + 1) - 1$.

Products II

- There exist products of horizontal Poisson forms [Kanatchikov '97]: Choose volume form $d^n x$ on $\mathcal{M}^n$, pull back to $\mathcal{P}$.
- Then every horizontal form f can be written as $f = i_{Z_f} d^n x$.
- Define $f \bullet g = i_{Z_g \wedge Z_f} d^n x$, satisfies Leibniz rule.
- This product is of scaling degree zero!
- Cannot extend to arbitrary Poisson forms without choice of a connection.

Conclusions

- The scaling degree is a powerful tool for the classification of Poisson forms.
- Its role has no analogue in mechanics.
- Can be used to correct the $\wedge$ product of Hamiltonian multi-vector fields.
- How to achieve a Leibniz rule is unclear: Problems with forms of scaling degree zero (pulled back from $\mathcal{E}$).
- Surprising: The results of this analysis can be stated without reference to coordinates, but the proofs depend heavily on the specific form of ω.