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Yang–Mills action from minimally coupled bosons on $\mathbb{R}^4$ and on the four-dimensional Moyal plane

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We consider bosons on (Euclidean) $\mathbb{R}^4$ that are minimally coupled to an external Yang–Mills field. We compute the logarithmically divergent part of the cutoff regularized quantum effective action of this system. We confirm the known result that this term is proportional to the Yang–Mills action. We use pseudodifferential operator methods throughout to prepare the ground for a generalization of our calculation to the noncommutative four-dimensional Moyal plane $\mathbb{R}^4_\theta$. We also include a detailed comparison of our cutoff regularization to heat kernel techniques. In the case of the noncommutative space, we complement the usual technique of asymptotic expansion in the momentum variable with operator theoretic arguments in order to keep separated quantum from noncommutativity effects. We show that the result from the commutative space $\mathbb{R}^4$ still holds if one replaces all pointwise products by the noncommutative Moyal product. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1839277]

I. INTRODUCTION

In this paper we study the determinant of certain differential operators. Such determinants naturally arise in quantum field theory at the one loop level. As the determinant of an operator on an infinite dimensional Hilbert space is not an *a priori* well-defined object, one must choose some regularization scheme. The latter means generally the choice of a recipe for how to replace the formal expressions by something that is both amenable to a rigorous definition and close in its properties. In our case of the regularization of determinants, a common starting point is the well-known identity

$$\log \det A = \operatorname{Tr} \log A, \tag{1}$$

which holds for (finite dimensional) matrices A. The task is now to give meaning to the trace on the right-hand side, since the operators of interest do not have a finite trace in general. In this paper, we restrict the trace to run over a subspace of our Hilbert space only. Loosely speaking, this subspace is spanned by wave functions that have a momentum expectation value smaller than a certain cutoff Λ. The precise definition will follow below. It is known that the cutoff regularized logarithm of the determinant, now viewed as a function of Λ, contains a term that scales like $\log \Lambda$ for large Λ. This term is closely related to the Wodzicki residue for the operator under consideration, a quantity that is of interest in the study of infinite dimensional geometry, see Ref. 15 for a recent review.

Motivated by the observation[14] that for fermions minimally coupled to an external Yang–Mills field, the logarithmically divergent part of the cutoff regularized logarithm of the determinant of the (massive) Dirac operator is proportional to the corresponding Yang–Mills action, we consider the case of bosons in an external Yang–Mills field on $\mathbb{R}^4$. With our work, we confirmed that the above result also applies to the bosonic case. The former result was proposed to be interpreted in two ways. On the one hand, the spectral action principle[2] states that the spectrum of the Dirac operator should provide exhaustive information about the complete bare action including the Yang–Mills expression. On the other hand, it is known that the logarithmically divergent part plays a critical role in the selection of the finite part of an effective action because of its behavior under rescaling of the regularization parameter Λ. From the latter viewpoint, it is desirable that the logarithmically divergent term in the regularized effective action produce expressions that occur in the complete bare action already.

To understand the connection between these two interpretations, it is interesting to consider the case of (scalar) bosons coupled to an external Yang–Mills field.

It is generally accepted that space–time might lose its smooth properties at very small scales. One possible mathematical framework for this is noncommutative geometry.[3] We are interested in a particular example, the four-dimensional (4D) Moyal plane,[10] also known as noncommutative flat space $\mathbb{R}^4_\theta$. Roughly speaking, the 4D Moyal plane differs from its Euclidean counterpart $\mathbb{R}^4$ in that there is an uncertainty relation for the simultaneous measurement of coordinates coming from the nonvanishing commutator

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu}.$$

Here x^μ, $\mu=1,\ldots,4$ are coordinates of $\mathbb{R}^4$ and Θ is some (antisymmetric) matrix. In this paper we take Θ to be proportional to the constant symplectic matrix, see Eq. (16). We refer to Ref. 5 for a treatment of Lorentz covariant generalization of this equation.

Although the results of our analysis for the case of bosons on the commutative space $\mathbb{R}^4$ are not new and can be found already in deWitt's book,[4] our consistent use of pseudodifferential operator methods technically makes possible the generalization to the noncommutative Moyal plane. We refer to Refs. 18 and 11 for a generalization of heat kernel regularization calculations to the noncommutative torus and the Moyal plane, respectively.

The remainder of this paper is structured as follows. Section II sets up the notation used in our work and states the results in the form of two propositions. Section III provides the necessary tools from the theory of pseudodifferential operators. The proofs of the statements from Sec. II can be found in Secs. IV and V, with detailed calculations postponed to the appendix. Also, Sec. IV contains additional arguments that make contact with the case of fermions on $\mathbb{R}^4$ and to an alternative regularization scheme, the heat kernel regularization. Section VI concludes with what we consider to be the lessons from our calculations.

II. NOTATION AND STATEMENT OF THE RESULTS

We consider the Klein–Gordon operator with minimally coupled external field on the four-dimensional flat Euclidean space $\mathbb{R}^4$, given by

$$\begin{aligned}\Box_A = D_A^\mu D_{A,\mu} = (\partial^\mu + ieA^\mu)(\partial_\mu + ieA_\mu) &= \partial^\mu\partial_\mu + ie\partial^\mu A_\mu + 2ieA^\mu\partial_\mu - e^2A^\mu A_\mu \\ &= \Box_0 + ie\partial^\mu A_\mu + 2ieA^\mu\partial_\mu - e^2A^\mu A_\mu. \end{aligned} \tag{2}$$

Here, $\mu=1,\ldots,4$ are the (Euclidean) indices of $\mathbb{R}^4$, $\partial_\mu=\partial/\partial x^\mu$, and A_μ are gl_N-valued Yang–Mills fields on $\mathbb{R}^4$. The bosonic wave functions are elements of the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^4) \otimes \mathbb{C}^N_{\text{color}},$$

where the last factor carries a gl_N representation from the external Yang–Mills fields. As an unbounded operator in $\mathcal{H}$, $\Box_0$ can be defined on smooth functions in $\mathcal{H}$ by its formal expression and then extended to a self-adjoint operator. We assume the Yang–Mills fields A_μ to be regular, i.e., to be smooth and to fall off (together with all their derivatives) at infinity like $|x|^{-2-\epsilon}$, $\epsilon>0$. The latter assumption ensures that all our spatial integrals below will converge. Also, for regular A_μ, the self-adjoint extension of $\Box_A$ can be computed from that of $\Box_0$. In what follows, we will not distinguish between the formal expression for $\Box_A$ and its self-adjoint extension.

We consider the cutoff regularized logarithm of the determinant of the massive Klein–Gordon operator

$$S_\Lambda(A) \coloneqq \mathrm{Tr}_\Lambda\left(\log\left(\frac{-\Box_A + m^2}{\Lambda_0^2}\right) - \log\left(\frac{-\Box_0 + m^2}{\Lambda_0^2}\right)\right), \tag{3}$$

where the cutoff regularized Hilbert space trace Tr_Λ sums over states with momentum bounded by Λ. More precisely, if D is an operator on $\mathcal{H}$ and Tr denotes the operator trace on $\mathcal{H}$, then the cutoff trace is defined by

$$\mathrm{Tr}_\Lambda\, D \coloneqq \mathrm{Tr}\{\theta(\Lambda^2 + \Box_0)D\}, \tag{4}$$

where θ is the Heaviside step function. As is well-known, the expression (3) occurs in quantum field theory as the one-loop effective action. Using the formal identity $\log\det = \mathrm{Tr}\log$, it can be viewed as the generalization of the determinant to operators on an infinite-dimensional Hilbert space.

The parameter Λ_0 has been introduced to balance physical dimensions. It also provides a useful tool for cross checking since in the result of our calculations, it should cancel. Moreover, in the definition of the regularized determinant, we have subtracted a term containing the free Klein–Gordon operator as a reference. Whereas this term is needed for turning the expression under the trace into a pseudodifferential operator, it also comes in—at least in the corresponding expression for fermions—when interpreting the determinant as a subsummation of the one loop diagrams in the Feynman path integral.[16]

The regularized determinant (3) has an asymptotic expansion in Λ for large values of Λ as

$$S_\Lambda(A) = c_2(A,m)\Lambda^2 + c_1(A,m)\Lambda^1 + c_{\log}(A,m)\log\Lambda + c_0(A,m) + \cdots, \tag{5}$$

where the dots indicate terms that vanish at least like $1/\Lambda$ in the limit $\Lambda \to \infty$.

We are interested in the coefficient $c_{\log}(A,m) \equiv c_{\log}(A)$.

Proposition II.1: For the regularized determinant $S_\Lambda(A)$ defined as above, the coefficient $c_{\log}(A,m)$ is proportional to the Yang–Mills action of A_μ,

$$c_{\log}(A) = \frac{1}{96\pi^2}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}_N(F^{\mu\nu}F_{\mu\nu}), \tag{6}$$

where tr_N is the matrix trace in gl_N and the curvature $F_{\mu\nu}$ of A_μ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu]. \tag{7}$$

The proof is contained in Sec. IV A. Note that the numerical factor in front of $F^{\mu\nu}F_{\mu\nu}$ differs from the one obtained in Ref. 4 Eqs. (24.16), etc., by $\frac{1}{2}$. By the considerations below, this can be understood as coming from the usage of a nongauge invariant regularization for $S_\Lambda(A)$. However, the latter allows a straightforward generalization to the noncommutative Moyal plane.

It is well-known that

$$(\not{D}_A)^2 = \mathbb{1}_4\Box_A - ie\sigma\cdot F, \tag{8}$$

where $\not{D}_A = \gamma^\mu(\partial_\mu + ieA_\mu)$, $\sigma\cdot F = \frac{1}{4}\sigma^{\mu\nu}F_{\mu\nu} = \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$, and γ^μ are the four-dimensional gamma matrices, i.e., 4×4 matrices that satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}_4, \tag{9}$$

$\eta^{\mu\nu}$ being the Euclidean flat metric. Using this identity, we are able to rederive the result of Ref. 14 concerning the determinant of the Dirac operator. This is demonstrated in Sec. IV B. Our computation has the advantage of avoiding extensive calculations involving the gamma matrices γ^μ.

It is at first sight surprising that the nongauge invariant definition of the determinant yields a gauge invariant logarithmically divergent part. It is therefore natural to consider the manifestly gauge invariant expression

$$\widetilde{S}_\Lambda(A) := \mathrm{Tr}_\Lambda^{\Box_A} \log\left(\frac{-\Box_A + m^2}{\Lambda_0^2}\right) - \mathrm{Tr}_\Lambda \log\left(\frac{-\Box_0 + m^2}{\Lambda_0^2}\right), \tag{10}$$

where in the first trace, the cutoff is taken with respect to the operator $\Box_A$ rather than $\Box_0$. As before, $\widetilde{S}_\Lambda(A)$ has an asymptotic expansion,

$$\widetilde{S}_\Lambda(A) = \widetilde{c}_2(A,m)\Lambda^2 + \widetilde{c}_1(A,m)\Lambda^1 + \widetilde{c}_{\log}(A,m)\log \Lambda + \widetilde{c}_0(A,m) + \cdots, \tag{11}$$

the dots subsuming terms scaling at least like $1/\Lambda$. A calculation in Sec. IV C reveals that the coefficient $\widetilde{c}_{\log}(A,m) \equiv \widetilde{c}_{\log}(A)$ in (11) equals half of the corresponding expression in $S_\Lambda(A)$,

$$\widetilde{c}_{\log}(A) = \tfrac{1}{2} c_{\log}(A). \tag{12}$$

This result agrees with the one obtained in Ref. 4.

A widely used alternative regularization of the determinant of a differential operator makes use of the ζ-function and the asymptotic expansion of the trace of the heat kernel operator. We want to compare our coefficient with earlier results that have been obtained with these methods[6] [see also the review articles Ref. 1, and references therein]. In this approach, one considers asymptotic expansions for the trace of the heat operator,

$$K(f,D) := \mathrm{Tr}_{L^2}(f e^{-tD}) = t^{-2} a_0(f,D) + t^{-3/2} a_1(f,D) + \cdots + a_4(f,D) + \cdots, \tag{13}$$

for small t, where f is some function on $\mathbb{R}^4$ that serves as a regulator for the spatial integrals. The rightmost dots indicate terms that fall off at least linearly in t. As the heat trace must be integrated on the positive axis together with the function t, the logarithmically divergent contribution to the heat kernel regularized trace is given by the coefficient $a_4(f,D)$. For the comparison of this coefficient to our result, let $c_{(\cdot)}(f,A)$, etc., be the coefficients in the expansion of $S_\Lambda(A)$, now spatially regularized in the same way as $K(f,D)$. In Sec. IV D, it is shown that the coefficient $\widetilde{c}_{\log}(f,A)$ in the asymptotic expansion of $\widetilde{S}_\Lambda(A)$ differs from the corresponding expression obtained via heat kernel regularization methods by a term proportional to m^4,

$$-\widetilde{c}_{\log}(f,A) + \frac{1}{32\pi^2} m^4 \int_{\mathbb{R}^4} \mathrm{d}^4x\, f(x) = a_4\left(f, \frac{-\Box_A + m^2}{\Lambda_0^2}\right). \tag{14}$$

The additional mass term can be traced back to the usage of the reference operator $-\Box_0 + m^2$ in (3). The calculations using the heat operator can be generalized to the noncommutative 4D torus[18] and the noncommutative Moyal plane.[11] The only change one encounters is that in all expressions, the commutative product of functions must be replaced by the noncommutative product $\star$.

The main part of our paper is devoted to the study of the case of the 4D Moyal plane as the underlying (noncommutative) "space." In this case, the algebra of functions on $\mathbb{R}^4$ is furnished with the (noncommutative) Moyal–Weyl product $\star := \star_\Theta$. The latter is defined by the integral formula

$$f \star g(x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \mathrm{d}^4y\, \mathrm{d}^4\xi\, e^{i\xi(x-y)} f(x - \tfrac{1}{2}\Theta\xi) g(y), \tag{15}$$

where Θ is a 4×4 matrix defined by

$$\Theta = \theta\begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix} \tag{16}$$

for the real parameter θ. In our calculations we do not use asymptotic expansions of this product in powers of θ.

On the Moyal plane, we consider the generalized Klein–Gordon operator

$$\Box_A^\theta = \partial^\mu\partial_\mu + ie(\partial^\mu A_\mu)\star + 2ieA^\mu \star \partial_\mu - e^2(A^\mu \star A_\mu)\star , \tag{17}$$

where $f\star$ is a short-hand notation for the operator that $\star$-multiplies smooth wave functions in $\mathcal{H}$ from the left by the (smooth) function f. We define $S_\Lambda^\theta(A)$ and $c_{(\cdot)}^\theta(A,m)$ in analogy with the formulas (3) and (5) above. Then, our main result is the following.

Proposition II.2: For minimally coupled bosonic fields on the (noncommutative) 4D Moyal plane, the above formula (6) holds with the commutative products replaced by the noncommutative Moyal–Weyl product, i.e., we have

$$c_{\log}^\theta(A) = \frac{1}{96\pi^2}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}_N F^{\theta,\mu\nu} \star F_{\mu\nu}^\theta, \tag{18}$$

where

$$F_{\mu\nu}^\theta = \partial_\mu A_\nu - \partial_\nu A_\mu + e[A_\mu, A_\nu]_\star. \tag{19}$$

III. PSEUDODIFFERENTIAL OPERATOR METHODS

In our work, we deal with a restricted class of *pseudodifferential operators* (ΨDO) which suits our purposes. The statements below may be found in Shubin's book.[17] We consider ΨDOs that act on smooth and compactly supported wave functions u as [$x=(x^\mu)$, $\mu=1,\ldots,4$, and likewise y, p describe points in $\mathbb{R}^4$; $xp=\Sigma_\mu x^\mu p^\mu$ denotes the scalar product, $|x|$ is the length of the vector x]

$$(Au)(x) = \int_{\mathbb{R}^4} \frac{\mathrm{d}^4p}{(2\pi)^4}\int_{\mathbb{R}^4} \mathrm{d}^4y\, \sigma[A](x,p)u(y)e^{ip(x-y)},$$

where the *symbol* $\sigma[A]$ of A is a smooth function that allows an *asymptotic expansion* in p according to

$$\sigma[A](x,p) \sim \sum_{r=0}^{\infty} \sigma_{m-r}[A](x,p).$$

Here, $\sim$ means that for each s, the finite sum $\Sigma_{r=0}^s \sigma_{m-r}[A]$ approximates $\sigma[A]$ up to a function that falls off at most as $|p|^{m-(s+1)}$ for large $|p|$,

$$\left|\partial_x^\alpha \partial_p^\beta\left(\sigma[A](x,p) - \sum_{r=0}^{s}\sigma_{m-r}[A](x,p)\right)\right| \leqslant C_{\alpha\beta}(1+|p|^2)^{[m-(s+1)-|\beta|]/2}$$

for all multi-indices $\alpha=(\alpha_1,\ldots,\alpha_4)$, $\beta=(\beta_1,\ldots,\beta_4)$, where

$$\partial_x^\alpha = \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x^4}\right)^{\alpha_4},$$

$|\alpha|=\alpha_1+\cdots+\alpha_4$, and $C_{\alpha\beta}$ are constants. The number m above is called the *order* of A. For a given symbol, there are many different asymptotic expansions. One particular choice is the asymptotic expansion in terms of *homogeneous symbols* $\sigma^h_{m-r}[A]$, i.e., smooth functions that in addition satisfy

$$\sigma^h_{m-r}[A](x,\lambda p) = \lambda^{m-r}\sigma^h_{m-r}[A](x,p) \quad \text{for } |p| = 1,\ \lambda > 1.$$

The first term $\sigma^h_m[A]$ in an asymptotic expansion in homogeneous summands is termed the *principal symbol*.

An asymptotic expansion encodes the information of a given symbol $\sigma[A]$ up to an additive function that falls off in p like a Schwartz test function. This piece of information will be sufficient for our purposes.

While the expansion in homogeneous symbols is appropriate to discuss invariant notions such as the residue of a ΨDO, the expansions obtained from recursion relations in the computation of resolvents of operators are not of this type in general. The two types, however, are related to each other through a finite resummation at every order of the infinite sum.

The action of the ΨDOs considered here can be extended to smooth functions, leading to the useful formula

$$\sigma[A](x,p) = e^{-ixp}Ae^{ixp}.$$

For the product AB of two ΨDOs A and B with respective symbols $\sigma[A]$ and $\sigma[B]$, one has the following asymptotic expansion of the symbol:

$$\sigma[A] * \sigma[B](x,p) = \sigma[AB](x,p) \sim \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!}\partial_p^\alpha \sigma[A](x,p)\partial_x^\alpha \sigma[B](x,p), \tag{20}$$

where the sum runs over all 4-indices α and we have used the notation $\alpha! = \alpha_1! \cdots \alpha_4!$. We will use $*$ whenever we mean this product of symbols, in contrast to the noncommutative product $\star$ defined later on.

Interpreting A as an operator in the Hilbert space $L^2(\mathbb{R}^4) \otimes \mathbb{C}^N$, we can compute the trace of A from its symbol according to

$$\mathrm{Tr}(A) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4}\int_{\mathbb{R}^4} \mathrm{d}^4x\ \mathrm{tr}_N\ \sigma[A](x,p),$$

where tr_N denotes the matrix trace over the gl_N-part of the symbol.

For operators A that do not have a (finite) trace, one considers the cutoff trace

$$\mathrm{Tr}_\Lambda(A) = \int_{|p|\leq\Lambda} \frac{\mathrm{d}^4p}{(2\pi)^4}\int_{\mathbb{R}^4} \mathrm{d}^4x\ \mathrm{tr}_N\ \sigma[A](x,p).$$

Clearly, this coincides with the previous definition of the cutoff regularized trace, Eq. (4).

The above expression has an asymptotic expansion in Λ, as can be seen from the asymptotic expansion of the symbol $\sigma[A]$ in homogeneous symbols. In this case, there appears a term scaling like $\log\Lambda$. On the other hand, the *Wodzicki residue*[19] of the operator A is defined as the angular p-integral and the spatial integral of the coefficient $\sigma^h_{-4}[A]$ in the homogeneous asymptotic expansion,

$$\mathrm{Res}(A) := \frac{1}{(2\pi)^4}\int_{|p|=1} \mathrm{d}\Omega_p \int_{\mathbb{R}^4} \mathrm{d}^4x\ \sigma^h_{-4}[A](x,p),$$

whenever the integral exists. It is known that for compact spatial manifolds this quantity determines completely the factor in front of the $\log\Lambda$ term in the asymptotic expansion of $\mathrm{Tr}_\Lambda(A)$. By abuse of notation, and motivated by the above observation, in our calculations we will use the expression $\mathrm{Res}(\cdots)$ to mean the factor in front of the $\log\Lambda$ term in the corresponding cutoff regularized trace.

IV. THE CASE $M=\mathbb{R}^4$

A. The logarithmically divergent part

In this section we compute the logarithmically divergent part of the bosonic effective action on $\mathbb{R}^4$. We define the regularized bosonic action as

$$S_\Lambda(A) := \mathrm{Tr}_\Lambda\left(\log\left(\frac{-\Box_A+m^2}{\Lambda_0^2}\right)-\log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right)\right). \tag{21}$$

We use the following expression for the logarithm:

$$\log(1+a)=\int_0^1\frac{\mathrm{d}s}{s}(1-(1+sa)^{-1}) \tag{22}$$

and recall the definition for the regularized trace of a pseudodifferential operator

$$\mathrm{Tr}_\Lambda(a) := \int_{|p|\le\Lambda}\frac{\mathrm{d}^4p}{(2\pi)^4}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,\sigma[a](x,p) \tag{23}$$

to get

$$\begin{aligned}\mathrm{Tr}_\Lambda&\left(\log\left(\frac{-\Box_A+m^2}{\Lambda_0^2}\right)-\log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right)\right)\\ &=-\int_{|p|\le\Lambda}\frac{\mathrm{d}^4p}{(2\pi)^4}\int_{\mathbb{R}^4}\mathrm{d}^4x\int_0^1\frac{\mathrm{d}s}{s}\mathrm{tr}_N\left(\sigma\left[\left(I+s\left(\frac{-\Box_A+m^2}{\Lambda_0^2}-I\right)\right)^{-1}\right]\right.\\ &\quad\left.-\sigma\left[\left(I+s\left(\frac{-\Box_0+m^2}{\Lambda_0^2}-I\right)\right)^{-1}\right]\right).\end{aligned} \tag{24}$$

As shown in the first section of the Appendix, the symbol of the resolvent of $\Box_A$ must satisfy the following recursion relation:

$$\sigma[(c_1I+c_2\Box_A)^{-1}](p,x)=\frac{1}{c_1-c_2p^2}-\frac{c_2}{c_1-c_2p^2}(\Box_A+2ip_\mu D_A^\mu)\sigma[(c_1I+c_2\Box_A)^{-1}](p,x).$$

Its formal solution is given by

$$\sigma[(c_1I+c_2\Box_A)^{-1}](x,p)=(c_1I+c_2(-p^2+\Box_A+2ip^\mu D_{A\mu}))^{-1}1,$$

which can be understood as defining an asymptotic expansion, see the Appendix for details. In particular, for our values of c_1 and c_2, we derive

$$\sigma\left[\left(I+s\left[\frac{-\Box_A+m^2}{\Lambda_0^2}-I\right]\right)^{-1}\right]\sim\sum_{n=0}^{\infty}\frac{(s/\Lambda_0^2)^n}{\left(1-s+\frac{sm^2}{\Lambda_0^2}+\frac{s}{\Lambda_0^2}p^2\right)^{n+1}}(\Box_A+2ip_\mu D_A^\mu)^n1.$$

Here and in all what follows, the 1 on the right-hand side (rhs) means that the operators $\Box_A$, $D_{A,\mu}$ should be applied to the N-dimensional constant vector.

Inserting this expansion into the integral and noting that the second symbol just cancels the first term in the expansion we then have

$$\mathrm{Tr}_\Lambda\left(\log\left(\frac{-\Box_A+m^2}{\Lambda_0^2}\right)-\log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right)\right)=-\int_{|p|\leqslant\Lambda}\frac{\mathrm{d}^4p}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{\Lambda_0^{2n}}\int_0^1\mathrm{d}s\frac{s^{n-1}}{\left(1+s\left(\frac{p^2+m^2}{\Lambda_0^2}-1\right)\right)^{n+1}}$$
$$\times\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N(\Box_A+2ip_\mu D_A^\mu)^n 1. \tag{25}$$

In the first section of the Appendix we will expand explicitly the terms in (25) and pick out the logarithmically diverging ones. Setting all the relevant terms together then gives

$$\mathrm{Res}\left(\log\left(\frac{-\Box_A+m^2}{\Lambda_0^2}\right)-\log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right)\right)$$
$$=-\frac{1}{8\pi^2}m^2\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\Box_A-\frac{1}{16\pi^2}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\Box_A^2+\frac{1}{8\pi^2}m^2\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\Box_A+\frac{1}{12\pi^2}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\Box_A^2$$
$$+\frac{1}{24\pi^2}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,D_A^\mu\Box_A D_{A\mu}-\frac{1}{48\pi^2}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N(\Box_A^2+D_A^\nu D_A^\mu D_{A\nu}D_{A\mu}+D_A^\mu\Box_A D_{A\mu})$$
$$=\frac{1}{48\pi^2}\left(\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,D_A^\mu\Box_A D_{A\mu}-\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,D_A^\nu D_A^\mu D_{A\nu}D_{A\mu}\right). \tag{26}$$

A short calculation shows that the terms under the trace are equal to $(e^2/2)F^{\mu\nu}F_{\mu\nu}$, so we finally get the result

$$\mathrm{Res}\left(\log\left(\frac{-\Box_A+m^2}{\Lambda_0^2}\right)-\log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right)\right)=\frac{e^2}{96\pi^2}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N F^{\mu\nu}F_{\mu\nu} \tag{27}$$

which proves Proposition II.1.

B. Comparison with fermion calculations

To incorporate fermions, we have to extend the Hilbert space. We take $\mathcal{H}_{\mathrm{fermion}}=L^2(\mathbb{R}^4)\otimes\mathbb{C}^N_{\mathrm{color}}\otimes\mathbb{C}^4_{\mathrm{spin}}$, where the last factor carries a representation of the Dirac gamma matrixes γ^μ, $\mu=1,\dots,4$.

We begin by computing the square of the Dirac operator $\not{D}$. First some definitions

$$D_{A\mu}=\partial_\mu+ieA_\mu,$$

$$\not{D}_A=\gamma^\mu(\partial_\mu+ieA_\mu).$$

A short calculation yields the well-known formula

$$(\not{D}_A)^2=\mathbb{1}_4\Box_A+\tfrac{1}{2}\gamma^\mu\gamma^\nu[D_{A\mu},D_{A\nu}]=\mathbb{1}_4\Box_A+ie\sigma\cdot F.$$

Here, $\mathbb{1}_4$ denotes the 4×4 unit matrix and

$$\sigma\cdot F:=\tfrac{1}{4}\sigma^{\mu\nu}F_{\mu\nu}=\tfrac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$$

for the matrices $\sigma^{\mu\nu}:=[\gamma^\mu,\gamma^\nu]$. We use the above identity to obtain

$$(-i\not{D}_A+im)(-i\not{D}_A-im)=-(\not{D}_A)^2+m^2=-\mathbb{1}_4\Box_A-ie\sigma\cdot F+m^2.$$

Taking the logarithm on both sides, for the left-hand side we arrive at

$$\log\left(\frac{-i\not{D}_A+im}{\Lambda_0}\right)\left(\frac{-i\not{D}_A-im}{\Lambda_0}\right)=\log\left(\frac{-i\not{D}_A+im}{\Lambda_0}\right)+\log\left(\frac{-i\not{D}_A-im}{\Lambda_0}\right),$$

while the right-hand side gives

$$\begin{aligned}\log(-\mathbb{1}_4\Box_A-ie\sigma\cdot F+m^2)&=\log\left(\frac{\mathbb{1}_4(-\Box_A+m^2)}{\Lambda_0^2}\right)\left(\mathbb{1}_4-\frac{ie}{-\Box_A+m^2}\sigma\cdot F\right)\\&=\log\left(\frac{\mathbb{1}_4(-\Box_A+m^2)}{\Lambda_0^2}\right)\\&\quad+\log\left(\mathbb{1}_4-\frac{ie}{-\Box_A+m^2}\sigma\cdot F\right)+\text{commutator terms}.\end{aligned}$$

The extra commutator terms can be computed from the Baker–Campbell–Hausdorff formula.

It is known that on compact manifolds the Wodzicki residue vanishes on commutators.[19] We therefore expect that from the above expression, the commutator terms will not contribute to the logarithmically divergent part of the regularized trace. In the second section of the Appendix it is shown explicitly that this is indeed the case. Rather than using integration-by-parts arguments, this is readily seen from the fact that the $\mathbb{C}_{\text{spin}}$-trace over $\sigma\cdot F$ gives zero. Also, the pertinent contributions from the first two terms of the right-hand side are calculated in the Appendix.

Furthermore, from Langmann's results[14] we know that $\text{Tr}_\Lambda\log[(-i\not{D}_A+im)/\Lambda_0]$ is independent of the sign of m, so we have

$$\begin{aligned}2\,\text{Tr}_\Lambda\log\left(\frac{-i\not{D}_A+im}{\Lambda_0}\right)&=4\,\text{Tr}_\Lambda\log\left(\frac{-\Box_A+m^2}{\Lambda_0^2}\right)+\frac{e^2}{16\pi^2}\log\Lambda\int_{\mathbb{R}^4}\mathrm{d}^4x\,\text{tr}(\sigma\cdot F)^2\\&\quad+\text{terms finite in }\Lambda,\end{aligned}$$

where the trace tr runs over both the $\mathbb{C}^N_{\text{color}}$ and the $\mathbb{C}^4_{\text{spin}}$ parts. Performing the trace over the γ-matrices yields

$$\text{tr}(\sigma\cdot F)^2=-2\text{tr}_N F^{\mu\nu}F_{\mu\nu}.$$

The result is then

$$\begin{aligned}\text{Tr}_\Lambda\log\left(\frac{-i\not{D}_A+im}{\Lambda_0}\right)&=2\,\text{Tr}_\Lambda\log\left(\frac{-\Box_A+m^2}{\Lambda_0^2}\right)-\frac{e^2}{16\pi^2}\log\Lambda\int_{\mathbb{R}^4}\mathrm{d}^4x\,\text{tr}_N\,F^{\mu\nu}F_{\mu\nu}+\cdots\\&=\left(\frac{e^2}{48\pi^2}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\text{tr}_N\,F^{\mu\nu}F_{\mu\nu}-\frac{e^2}{16\pi^2}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\text{tr}_N\,F^{\mu\nu}F_{\mu\nu}\right)\log\Lambda+\cdots\\&=-\frac{e^2}{24\pi^2}\log\Lambda\int_{\mathbb{R}^4}\mathrm{d}^4x\,\text{tr}_N\,F^{\mu\nu}F_{\mu\nu}+\text{terms finite in }\Lambda,\end{aligned}$$

in agreement with Ref. 14.

C. Dependence on the regularization scheme

So far we have been looking at the cutoff regularized determinant

$$S_\Lambda(A) = \mathrm{Tr}_\Lambda\left(\log\left(\frac{-\Box_A + m^2}{\Lambda_0^2}\right) - \log\left(\frac{-\Box_0 + m^2}{\Lambda_0^2}\right)\right). \quad (28)$$

As the cutoff in this regularization is taken with respect to the reference operator $\Box_0$, the above expression is not manifestly gauge invariant. It is thus surprising that the coefficient $c_{\log}(A)$ turns out to be gauge invariant.

One could use the spectral projection with respect to $\Box_A$ instead, but again the resulting expression would fail to be manifestly gauge invariant now because of the reference term $\log[(-\Box_0+m^2)/\Lambda_0^2]$. The latter had to be included to make the calculations tractable by the methods of classical ΨDOs.

Of course, there are gauge invariant regularization schemes such as heat kernel regularization (see the review articles in Ref. 1 for recent developments in this field) readily available. However, cutoff regularized traces seem to be closer to physical intuition.

An acceptable, manifestly gauge invariant expression would be

$$\tilde{S}_\Lambda(A) \coloneqq \mathrm{Tr}_\Lambda^{\Box_A} \log\left(\frac{-\Box_A + m^2}{\Lambda_0^2}\right) - \mathrm{Tr}_\Lambda \log\left(\frac{-\Box_0 + m^2}{\Lambda_0^2}\right), \quad (29)$$

where

$$\mathrm{Tr}_\Lambda^{\Box_A} \log\left(\frac{-\Box_A + m^2}{\Lambda_0^2}\right) \coloneqq \mathrm{Tr}\left\{P_\Lambda(\Box_A)\log\left(\frac{-\Box_A + m^2}{\Lambda_0^2}\right)\right\}$$

is defined using the spectral projections $P_\Lambda(\Box_A) \coloneqq \theta(\Lambda^2 - \Box_A)$ of $\Box_A$, where θ denotes the Heaviside step function that is zero for negative arguments and equal to 1 otherwise. It turns out that $\tilde{S}_\Lambda(A)$ has an asymptotic expansion as

$$\tilde{S}_\Lambda(A) = \tilde{c}_2(A)\Lambda^2 + \tilde{c}_1(A)\Lambda + \tilde{c}_{\log}(A)\log\Lambda + \cdots . \quad (30)$$

The dots indicate terms that are finite in the large Λ limit.

In this section we want to compare the coefficient $\tilde{c}_{\log}(A)$ of the logarithmically divergent part in the above expression to the coefficient $c_{\log}(A)$ computed earlier.

A short calculation reveals how to proceed,

$$\begin{aligned}
&\mathrm{Tr}_\Lambda^{\Box_A} \log \frac{-\Box_A + m^2}{\Lambda_0^2} - \mathrm{Tr}_\Lambda^{\Box_0} \log \frac{-\Box_0 + m^2}{\Lambda_0^2} \\
&\quad = \mathrm{Tr}_\Lambda^{\Box_0}\left(\log \frac{-\Box_A + m^2}{\Lambda_0^2} - \log \frac{-\Box_0 + m^2}{\Lambda_0^2}\right) + (\mathrm{Tr}_\Lambda^{\Box_A} - \mathrm{Tr}_\Lambda^{\Box_0})\left(\log \frac{-\Box_A + m^2}{\Lambda_0^2} - \log \frac{-\Box_0 + m^2}{\Lambda_0^2}\right) \\
&\qquad + (\mathrm{Tr}_\Lambda^{\Box_A} - \mathrm{Tr}_\Lambda^{\Box_0})\log \frac{-\Box_0 + m^2}{\Lambda_0^2}. \quad (31)
\end{aligned}$$

Obviously, the coefficient $\tilde{c}_{\log}(A)$ receives contributions from three different terms, only the first of which is given by $c_{\log}(A)$. From the calculation of the pertinent part in the third term, it will be apparent that the second one in fact does not contribute to $\tilde{c}_{\log}(A)$. For the computation of the third term in (31), however, we must introduce an additional regulator that deals with the noncompactness of $\mathbb{R}^4$. Let f be a smooth, compactly supported function on $\mathbb{R}^4$, interpreted as a multiplication operator on $\mathcal{H}$. Then

$$\mathrm{Tr}\left\{ f\theta(\Lambda^2+\Box_A)\log\frac{-\Box_0+m^2}{\Lambda_0^2}\right\} = \int_{\mathbb{R}^4}\frac{\mathrm{d}^4p}{(2\pi)^4}\int_{\mathbb{R}^4} f(x)\sigma[\theta(\Lambda^2+\Box_A)]\sigma\left[\log\frac{-\Box_0+m^2}{\Lambda_0^2}\right] + \cdots . \tag{32}$$

The dots indicate contributions from the star product of symbols that are uniformly bounded in Λ. Use has been made of the fact that f and $\sigma[\log(-\Box_0+m^2)/\Lambda_0^2]$ are independent of p and x, respectively.

As a next step, we need to derive an asymptotic expansion for the symbol of the theta function. We start with the following sum expression for a smooth approximation of the Heaviside θ function (Ref. 9, p. 248, etc.):

$$\theta_\epsilon(x) = \frac{1}{\epsilon}\sum_{r=-\infty}^{\infty}\frac{e^{i\omega_r 0^+}}{x+i\omega_r} = \frac{e^{-x0^+}}{e^{-x\epsilon}+1}, \quad \omega_r = (2r+1)\pi/\epsilon, \tag{33}$$

for $\epsilon > 0$. The step function is regained in the limit $\epsilon \to \infty$. Using this equation, we derive an asymptotic expansion for the symbol of $\theta(\Lambda^2+\Box_A)$ as (for details we refer to the third section of the Appendix)

$$\sigma[\theta_\epsilon(\Lambda^2+\Box_A)] = \frac{1}{2\pi i}\int \mathrm{d}z\, e^{iz\epsilon}\sigma\left[\frac{1}{z-i(\Lambda^2+\Box_A)}\right] \sim \sum_{n=0}^{\infty}\frac{1}{n!}\delta_\epsilon^{(n-1)}(\Lambda^2-p^2)(\Box_A+2ip^\mu D_{A\mu})1.$$

As before, for mnemonic purposes, this asymptotic series can be summarized as

$$\sigma[\theta(\Lambda^2+\Box_A)](x,p) = \theta(\Lambda^2-p^2+\Box_A+2ip^\mu D_{A\mu})1, \tag{34}$$

where the x dependence originates from the external fields A.

Combining our results, we find

$$\mathrm{Tr}\left\{ f\theta(\Lambda^2+\Box_A)\log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right)\right\} - \mathrm{Tr}\left\{ f\theta(\Lambda^2+\Box_0)\log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right)\right\}$$
$$= \sum_{n=1}^{\infty}\frac{1}{n!}\int_{\mathbb{R}^4}\frac{\mathrm{d}^4p}{(2\pi)^4}\int_{\mathbb{R}^4}\mathrm{d}^4x\, f(x)\,\delta_\epsilon^{(n-1)}(\Lambda^2-p^2)\log\left(\frac{p^2+m^2}{\Lambda_0^2}\right)\mathrm{tr}_N\{(\Box_A+2ip^\mu D_{A\mu})^n 1\}.$$

Obviously, we can now drop the regulator f.

For large Λ, the δ_ϵ-functions cancel the radial p-integration. Therefore, the only contributions to the logarithmically divergent part in the above expression can originate from terms where the derivatives of the δ_ϵ-functions exclusively hit the trace under the integral of the measure d^4p but not the factor $\log[(-\Box_0+m^2)/\Lambda_0^2]$. This is only possible as long as $2(n-1)\leqslant 3+n$ (the derivatives of δ_ϵ count twice because of the p^2 in the argument, and the 3 on the rhs comes from the measure d^4p) and hence $n\leqslant 5$. Moreover, since the angular p-integration over an odd number of factors p^μ gives always zero, the $n=5$ term cannot contribute either.

As shown in the Appendix, we can now expand the powers $(\Box_A+2ip^\mu D_{A\mu})^n 1$ for $n\leqslant 4$, perform the angular p-integrations, substitute $p^2\to u$ and use partial integration to get rid of the derivatives of the δ_ϵ-functions. We arrive at

$$
\begin{aligned}
&\mathrm{Tr}_\Lambda^{\Box_A}\log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right)-\mathrm{Tr}_\Lambda^{\Box_0}\log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right)\\
&\quad=\frac{1}{16\pi^2}\int_1^\infty \mathrm{d}u\,\log\left(\frac{u+m^2}{\Lambda_0^2}\right)\delta_\epsilon(\Lambda^2-u)u(1-1)\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\Box_A 1+\frac{1}{16\pi^2}\int_1^\infty \mathrm{d}u\,\log\left(\frac{u+m^2}{\Lambda_0^2}\right)\\
&\qquad\times\delta_\epsilon(\Lambda^2-u)\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{6}\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\mathrm{tr}_N\Box_A^2 1+\frac{1}{16\pi^2}\int_1^\infty \mathrm{d}u\,\log\left(\frac{u+m^2}{\Lambda_0^2}\right)\delta_\epsilon(\Lambda^2-u)(-\tfrac{1}{3}+\tfrac{1}{6})\\
&\qquad\times\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,D_A^\mu\Box_A D_{A\mu}1+\frac{1}{16\pi^2}\int_1^\infty \mathrm{d}u\,\log\left(\frac{u+m^2}{\Lambda_0^2}\right)\frac{1}{6}\delta_\epsilon(\Lambda^2-u)\\
&\qquad\times\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,D_A^\mu D_A^\nu D_{A\mu}D_{A\nu}1+\cdots\\
&\quad=-\frac{1}{96\pi^2}\log\left(\frac{\Lambda^2+m^2}{\Lambda_0^2}\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,D_A^\mu\Box_A D_{A\mu}1\\
&\qquad+\frac{1}{96\pi^2}\log\left(\frac{\Lambda^2+m^2}{\Lambda_0^2}\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N D_A^\mu D_A^\nu D_{A\mu}D_{A\nu}1+\cdots\\
&\quad=-\frac{1}{2}\frac{1}{96\pi^2}\log\left(\frac{\Lambda^2+m^2}{\Lambda_0^2}\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,F^{\mu\nu}F_{\mu\nu}+\cdots\ ,
\end{aligned}
$$

where the dots indicate finite or polynomially divergent contributions.

Finally, we will turn back to the second term in (31). The difference as compared to the previous calculation is that now the symbol of the operator under the traces has an asymptotic expansion that is a power series in $1/p$. Therefore, in contrast to the above, no logarithmically divergent term will occur in a large Λ expansion.

Combining these results with our previous expression for $c_{\log}(A)$, we find

$$\tilde{c}_{\log}(A)=\frac{1}{2}\frac{1}{96\pi^2}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,F^{\mu\nu}F_{\mu\nu}=\frac{1}{2}c_{\log}(A). \tag{35}$$

D. Comparison with heat kernel regularization

In this section we want to compare our results with previous ones in the literature[18,11] obtained by heat kernel techniques. For a given differential operator D, we consider the trace of the heat operator for D,

$$K(t,f,D)=\mathrm{Tr}(fe^{-tD}),$$

where the auxiliary smooth function $f(x)$ is introduced to make spatial integrals converge on $\mathbb{R}^n$. We write the effective action for D as

$$S=-\int_0^\infty \frac{\mathrm{d}t}{t}K(t,f,D).$$

Here the formula $\log\det(D)=\mathrm{Tr}\log(D)$ has been used again together with the following formal expression for the logarithm:

$$\log\lambda = -\int_0^\infty \frac{\mathrm{d}t}{t}e^{-t\lambda},$$

which holds up to an (infinite) integration constant.

There is an asymptotic expansion for the heat trace as $t\to 0$ given by

$$\mathrm{Tr}(fe^{-tD}) \sim \sum_{k\geqslant 0} t^{(k-n)/2}a_k(f,D).$$

Next we define the ζ-function for D as follows:

$$\zeta(s,f,D) = \mathrm{Tr}(fD^{-s}).$$

Writing the ζ-function in terms of the heat trace as

$$\zeta(s,f,D) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\, t^{s-1}K(t,f,D),$$

we see that $\Gamma(s)\zeta(s,f,D)$ has simple poles at the points $s=(n-k)/2$ and the complex residue at $s=(n-k)/2$ is given by

$$\mathrm{Res}_{s=(n-k)/2}(\Gamma(s)\zeta(s,f,D)) = a_k(f,D). \tag{36}$$

From the asymptotic expansion of the heat trace and the integral formula for the effective action S we see that the logarithmically divergent part is given when $k=n$ so we are interested in computing the coefficient $a_n(f,(-\Box_A+m^2)/\Lambda_0^2)$. In our case $n=4$.

The first task is to compute the ζ-function for the operator $(-\Box_A+m^2)/\Lambda_0^2$. From the definition of the ζ-function we have

$$\zeta\left(s,f,\frac{-\Box_A+m^2}{\Lambda_0^2}\right) = \int_{\mathbb{R}^4}\frac{\mathrm{d}^4p}{(2\pi)^4}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,\sigma[f]*\sigma\left[\left(\frac{-\Box_A+m^2}{\Lambda_0^2}\right)^{-s}\right](x,p).$$

We next use the expansion

$$(a+x)^{-s} = \sum_{r=0}^{\infty}(-1)^r\frac{\Gamma(s+r)}{r!\Gamma(s)}a^{-(r+s)}x^r$$

to write the symbol of $(-\Box_A+m^2)/\Lambda_0^2$ as

$$\sigma\left[\left(\frac{-\Box_A+m^2}{\Lambda_0^2}\right)^{-s}\right] = \left(\frac{p^2+m^2-\Box_A-2ip^\mu D_{A\mu}}{\Lambda_0^2}\right)^{-s}$$
$$\sim \sum_{r=0}^{\infty}(-1)^r\frac{\Gamma(s+r)}{r!\Gamma(s)}\Lambda_0^{2s}\frac{1}{p^{2(s+r)}}(m^2-\Box_A-2ip^\mu D_{A\mu})^r 1.$$

Splitting the integration in the ζ-function into two parts we then have

$$\zeta\left(s,f,\frac{-\Box_A+m^2}{\Lambda_0^2}\right) = \int_{|p|\leqslant 1}\frac{\mathrm{d}^4p}{(2\pi)^4}\int_{\mathbb{R}^4}\mathrm{d}^4x\, f(x)\mathrm{tr}_N\,\sigma\left[\left(\frac{-\Box_A+m^2}{\Lambda_0^2}\right)^{-s}\right](x,p)$$
$$+\sum_{r=0}^{\infty}\frac{(-1)^r\,\Gamma(s+r)}{r!\Gamma(s)}\Lambda_0^{2s}\int_1^\infty |p|^3\mathrm{d}|p|\frac{1}{p^{2(r+s)}}\int_{S^3}\frac{\mathrm{d}\Omega_p}{(2\pi)^4}\int_{\mathbb{R}^4}\mathrm{d}^4x\, f(x)$$
$$\times\mathrm{tr}_N(m^2-\Box_A-2ip^\mu D_{A\mu})^r 1. \tag{37}$$

Using the fact that under the angular integration odd powers of p give zero we can write

$$\int_{S^3}\frac{d\Omega_\xi}{(2\pi)^4}\int_{\mathbb{R}^4} d^4x\, f(x)\mathrm{tr}_N(m^2-\Box_A-2ip^\mu D_{A\mu})^r 1=\sum_{k=0}^{[r/2]}(-2i)^{2k}p^{2k}\,\mathrm{d}(f,r,2k)$$

for some functions $\mathrm{d}(f,r,2k)$ of f, $\Box_A$ and $D_{A\mu}$ determined from the expansion of $(m^2-\Box_A-2ip^\mu D_{A\mu})^r$. In particular, we have

$$\mathrm{d}(f,0,0)=\frac{1}{8\pi^2}\int_{\mathbb{R}^4} d^4x\, f(x).$$

We then find

$$\zeta\left(s,f,\frac{-\Box_A+m^2}{\Lambda_0^2}\right)=\chi(s)+\sum_{r=0}^{\infty}\frac{(-1)^r\Lambda_0^{2s}\Gamma(s+r)}{r!\Gamma(s)}\int_1^\infty |p|^3\, d|p|\frac{1}{p^{2(r+s)}}\sum_{t=0}^{[r/2]}(-2i)^{2t}p^{2t}\,\mathrm{d}(f,r,2t),$$

where $\chi(s)$ denotes the first integral in the rhs of (37), a holomorphic function in s. We can evaluate explicitly the p-integral in the above expression to obtain the following formula for the ζ-function:

$$\zeta\left(s,f,\frac{-\Box_A+m^2}{\Lambda_0^2}\right)=\chi(s)+\frac{\Lambda_0^{2s}}{\Gamma(s)}\sum_{r=0}^{\infty}\sum_{t=0}^{[r/2]}\frac{1}{2}\frac{(-1)^{r+t}4^t\Gamma(s+r)}{r!}\frac{1}{s-(2-r+t)}\mathrm{d}(f,r,2t). \tag{38}$$

There are two parts of the ζ-function contributing to the residue at $s=0$; the gamma function $\Gamma(s+r)$ and the poles of $1/[s-(2-r+t)]$. The first one gives a contribution for $r=0$ and the latter one when $r=2+t$. From the summation we see that $t\leqslant r/2$ so it follows that only the terms with $t\leqslant 2$ contribute to the residue:

$$\begin{aligned}a_4\left(f,\frac{-\Box_A+m^2}{\Lambda_0^2}\right)&=\mathrm{Res}_{s=0}\,\Gamma(s)\zeta\left(s,f,\frac{-\Box_A+m^2}{\Lambda_0^2}\right)\\&=\chi(0)-\frac{1}{4}\mathrm{d}(f,0,0)+\sum_{t=0}^{2}\frac{4^t\Gamma(2+t)}{(2+t)!}\frac{1}{2}\mathrm{d}(f,t+2,2t),\end{aligned}$$

where $\chi(0)$ is given by

$$\chi(0)=\frac{1}{(2\pi)^4}\int_{|p|\leqslant 1} d^4p\int_{\mathbb{R}^4} d^4x\, f(x)=\int_0^1 |p|^3\, d|p|\mathrm{d}(f,0,0)=\frac{1}{4}\mathrm{d}(f,0,0).$$

We have thus obtained the following expression for $a_4(f,(-\Box_A+m^2)/\Lambda_0^2)$:

$$a_4\left(f,\frac{-\Box_A+m^2}{\Lambda_0^2}\right)=\sum_{t=0}^{2}\frac{1}{2}\frac{4^t\Gamma(2+t)}{(2+t)!}\mathrm{d}(f,t+2,2t).$$

We now compute directly the logarithmically divergent part of $S_\Lambda(A)$. For this we need the following formula:

$$\sigma\left[\log\left(\frac{-\Box_A+m^2}{\Lambda_0^2}\right)-\log\left(\frac{-\Box+m^2}{\Lambda_0^2}\right)\right]=\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}\frac{1}{p^{2r}}[(m^2-\Box_A-2ip^\mu D_{A\mu})^r-m^{2r}].$$

Note that for this asymptotic expansion, we have divided the recursion formula (A8) differently.

We split the p-integration in the trace into two parts to get

$$\mathrm{Tr}_\Lambda \log\left(f\left(\frac{-\Box_A+m^2}{\Lambda_0^2}-\frac{-\Box_0+m^2}{\Lambda_0^2}\right)\right)$$

$$= \text{finite terms in } \Lambda + \sum_{r=1}^{\infty}\sum_{t=0}^{[r/2]}\frac{(-1)^{r+1}}{r}\int_1^\Lambda |p|^3\, \mathrm{d}|p| \frac{1}{p^{2(r-t)}}(-1)^t 4^t\, \mathrm{d}(f,r,2t)$$

$$-\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}m^{2r}\int_1^\Lambda |p|^3\, \mathrm{d}|p|\frac{1}{p^{2r}}\mathrm{d}(f,0,0).$$

The logarithmically divergent part is then given by

$$c_{\log}(A) = -\sum_{t=0}^{2} 4^t \frac{1}{t+2}\mathrm{d}(f,t+2,2t) + \frac{1}{2}m^4\, \mathrm{d}(f,0,0)$$

so we finally have the result

$$-\frac{1}{2}c_{\log}(A) + \frac{1}{4}m^4\, \mathrm{d}(f,0,0) = a_4(f,(-\Box_A+m^2)/\Lambda_0^2).$$

Remarks: (1) The coefficients $\mathrm{d}(f,r,2k)$ defined below Eq. (37) are given by spatial integrals over the $gl(N)$-trace of certain polyomials in the external fields and their derivatives. They can be easily computed by expanding the power on the left-hand side of the defining formula, using the well-known expressions for the angular p-integration of polynomials in p^μ.

(2) Note that the argument relating a_{-4} and $c_{\log}$ did not use the specific form of the coefficients $\mathrm{d}(f,r,2k)$. Therefore, it can be extended to a larger class of operators.

(3) Combining Eqs. (36) and (38), we have a formula for the calculation of the coefficients a_k at hand. In particular, evaluating the function $\chi(s)$ for negative integer s amounts to the computation of the symbol of $(-\Box_A+m^2)^l$ for positive integer powers of l. The latter can be obtained from the formula

$$\sigma[(-\Box_A+m^2)^{l+1}] = (p^2+m^2-\Box_A - 2ip_\mu D_A^\mu)\sigma[(-\Box_A+m^2)^l]$$

and the symbol of $-\Box_A+m^2$.

V. GENERALIZATION TO THE MOYAL PLANE

A. The Moyal plane $\mathbb{R}^4_\theta$: generalities

In this section, we want to replace the manifold $\mathbb{R}^4$ by the four-dimensional Moyal plane $\mathbb{R}^4_\theta$, an example of a noncommutative manifold.

For the definition of the latter, one must specify (among other things; see Ref. 3 for the general theory, Ref. 10 for the treatment of the Moyal plane in this context) a (noncommutative) associative algebra $\mathcal{A}$, the elements of which generalize the notion of (smooth) functions on an ordinary manifold. In the case of the 4D Moyal plane, the algebra $\mathcal{A}$ is taken to include the rapidly decaying Schwartz test functions on $\mathbb{R}^4$, while the product of two such elements f, g is given by the integral formula

$$(f\star g)(x) = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\mathrm{d}^4y\, \mathrm{d}^4\xi\, e^{i\xi(x-y)}f(x-\tfrac{1}{2}\Theta\xi)g(y), \tag{39}$$

where Θ is a 4×4 matrix defined by

$$\Theta = \theta\begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0\end{pmatrix} \tag{40}$$

for the real parameter θ.

The elements of $\mathcal{A}$ act on the Hilbert space $L^2(\mathbb{R}^4)$ by left $\star$-multiplication (see Ref. 8 for an extension of the above formula to distributions). For an element $f \in \mathcal{A}$, we will write the corresponding operator on $L^2(\mathbb{R}^4)$ as $f\star$. From the integral formula (39) of $\star$, we can see that $f\star$ is a ΨDO with the symbol

$$\sigma[f\star](x,p) = f(x - \tfrac{1}{2}\Theta p). \tag{41}$$

Note that the asymptotic behavior of f is transferred to the p dependence of the symbol of $f\star$. In particular, for rapidly decaying f, $f\star$ is infinitely smoothing.[10]

A natural class of functions suitable for the Moyal product is the set $\mathcal{P}$ of infinitely differentiable functions f on $\mathbb{R}^4$ such that, for a real number s and for every multi-index α,

$$|(\partial_x^\alpha f)(x)| \leqslant C_\alpha (1 + x^2)^{(s-|\alpha|)/2}, \tag{42}$$

s is called the *order* of f. For $f, g \in \mathcal{P}$ and of order s_1, s_2, respectively, $f\star g$ is again in $\mathcal{P}$ and of order s_1+s_2 (Ref. 12, Sect. 7).

B. Calculation of the logarithmically divergent part

With the commutative product of functions on $\mathbb{R}^4$ replaced by the Moyal product $\star$, Eq. (39), we are led to study the following variant of the Klein–Gordon operator

$$\Box_A^\theta \psi = \partial^\mu \partial_\mu \psi + ie(\partial^\mu A_\mu) \star \psi + 2ieA^\mu \star \partial_\mu \psi - e^2 A^\mu \star A_\mu \star \psi$$

for any rapidly decaying smooth function ψ in the Hilbert space $\mathcal{H}=L^2(\mathbb{R}^4)\otimes \mathbb{C}^N_{\text{color}}$. Here, the matrix valued Yang–Mills fields A_μ are taken to be in the set $\mathcal{P}$ above with order strictly smaller than -4, i.e., to satisfy (42) with $s<-4$.

We will also need the operator $D_{A\mu}^\theta$, defined by

$$D_{A\mu}^\theta \psi = \partial_\mu \psi + ieA_\mu \star \psi, \quad \psi \in \mathcal{S}(\mathbb{R}^4).$$

In analogy with the first section, we consider the cutoff regularized determinant of $(-\Box_A^\theta + m^2)/\Lambda_0^2$,

$$S_\Lambda^\theta(A) := \mathrm{Tr}_\Lambda \left\{ \log \frac{-\Box_A^\theta + m^2}{\Lambda_0^2} - \log \frac{-\Box_0 + m^2}{\Lambda_0^2} \right\}.$$

As before, the trace will be computed from the symbol of $\log[(-\Box_A^\theta + m^2)/\Lambda_0^2]$. For the latter, we will need an expression for the symbol of the resolvent of $\Box_A^\theta$. Again, this will be obtained via a recursion relation.

As explained in the Appendix, we find for $c_1, c_2 \in \mathbb{C}$, $c_1 \cdot c_2 < 0$ or $c_2=0$,

$$\sigma[(c_1 + c_2\Box_A^\theta)^{-1}](x,p) = \frac{1}{c_1 - c_2 p^2} - \frac{c_2}{c_1 - c_2 p^2}(\Box^\theta_{A(\cdot-(1/2)\Theta p)} + 2ip^\mu D^\theta_{A(\cdot-(1/2)\Theta p)})\sigma[(c_1 + c_2\Box_A^\theta)^{-1}] \times (x,p),$$

where $A(\cdot - \frac{1}{2}\Theta p)$ is a short-hand notation for the external fields A_μ shifted by $-\frac{1}{2}\Theta p$ in their argument, $A(\cdot - \frac{1}{2}\Theta p)(x) = A(x - \frac{1}{2}\Theta p)$. In the derivation of the recursion relation, we have used the identity[13]

$$e^{ip(x-y)} f(x) = [f(\cdot + \tfrac{1}{2}\theta) \star e^{ip(\cdot - y)}](x)$$

and associativity of the Moyal product.

From the recursion relation, one readily obtains the formal expression

$$\sigma[(c_1 + c_2\Box_A^\theta)^{-1}](x,p) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{(c_1 - c_2 p^2)^{n+1}} (\Box^\theta_{A(\cdot-(1/2)\Theta p)} + 2ip^\mu D^\theta_{A(\cdot-(1/2)\Theta p)})^n 1.$$

A thorough investigation reveals, however, that an interpretation of this equation as an asymptotic expansion in p would be misleading: The p dependence through the arguments of the external fields $A_\mu(x-\frac{1}{2}\Theta p)$ is superficial in that it goes away under the spatial integral. Therefore, one must develop different tools to tackle the situation. As shown in the Appendix, the operator R_N defined by the sum of the first N terms in the above series, for N sufficiently large, differs from the operator $(c_1+c_2\Box_A^\theta)^{-1}$ by a trace-class operator only. Hence, for the singular behavior of the cutoff regularized trace, it suffices to consider this operator R_N.

Inserting the expression for the symbol of R_N into the integral formula for the logarithm, Eq. (22), we find

$$S_\Lambda^\theta(A) = -\sum_{n=1}^{N} \int_{|p|\le\Lambda} \frac{\mathrm{d}^4 p}{(2\pi)^4} \int_0^1 \frac{\mathrm{d}s}{s} \frac{s^{n-1}}{\left(1 + s\left(\frac{p^2+m^2}{\Lambda_0^2} - 1\right)\right)^{n+1}} \int_{\mathbb{R}^4} \mathrm{d}^4 x \, \mathrm{tr}_N (\Box^\theta_{A(\cdot-(1/2)\Theta p)} + 2ip^\mu D^\theta_{A(\cdot-(1/2)\Theta p),\mu})^n 1 + \text{terms finite in } \Lambda.$$

Now, for every term in the sum, we can shift the x-integration by $-\frac{1}{2}\Theta p$. After this substitution the contribution to the Λ-behavior is apparent: It is only the first four terms that can contribute to $c_{\log}^\theta(A)$. Moreover, the resulting expression differs from the corresponding $S_\Lambda(A)$, Eq. (25), solely in the appearance of the product $\star$ in place of the commutative product. As the replacement of the latter by the Moyal product does not affect the asymptotic behavior in the variable p, we conclude

$$c_{\log}^\theta(A) = \frac{e^2}{96\pi^2} \int_{\mathbb{R}^4} \mathrm{d}^4 x \, \mathrm{tr}_N \, F^{\theta,\mu\nu} \star F^\theta_{\mu\nu}, \tag{43}$$

where $F^\theta_{\mu\nu}$ is defined by

$$F^\theta_{\mu\nu} = -ie[D^\theta_{A\mu}, D^\theta_{A\nu}].$$

This proves the claim of Proposition II.2.

VI. CONCLUSION

In the first part of our paper, we considered the regularized determinant of the Klein–Gordon operator $\Box_A$ with minimal coupling on $\mathbb{R}^4$. For the regularization, we restricted the Hilbert space trace to run over states of momentum below some cutoff Λ.

Although similar results have been obtained before, we choose to present here an approach that consistently uses the pseudodifferential operator methods to prepare the ground for calculations on a particular noncommutative manifold.

A useful formula for the calculations with symbols of pseudodifferential operators (ΨDO) is given by

$$\sigma[f(\Box_A)](x,p) = f(-p^2 + \Box_A + 2ip^\mu D_{A\mu})1 \tag{44}$$

for any function f of the Klein–Gordon operator $\Box_A$. This formula originates from a recursion relation for the symbol $\sigma[f(\Box_A)]$. It is to be understood as defining an asymptotic expansion of the symbol for large p.

Using this asymptotic expansion we could indeed confirm that the cutoff regularized trace does have an asymptotic expansion in the cutoff Λ as in Eq. (5). Although our approach did not use a manifestly gauge invariant regularization, the term scaling like $\log\Lambda$ in the regularized trace of the logarithm of the massive Klein–Gordon operator was found to be gauge invariant. However, the numerical coefficient in front of this expression differs from that obtained via manifestly gauge invariant methods[4,11] by a factor of −2, see Eqs. (12) and (14). This difference can be verified through a comparison of our approach to heat kernel regularization. It turns out that this argument does not rely on the particular structure of the operator $\square_A$, cf. the use of the functions $\mathrm{d}(f,r,2t)$ in Sec. IV D, so we expect it to hold even for more general operators as well. It would be interesting to understand this feature in more detail. Also, we propose a gauge invariant version of the cutoff regularization, Eq. (10), which reproduces the result of Refs. 4 and 11.

Recently, zeta functions have been found to show a pole structure on noncommutative torus[7] that differs from the commutative case. It would be interesting to see a similar effect for the Moyal plane by means of the development in Sec. IV D.

In the second and main part of our work, we considered the generalized Klein–Gordon operator for minimally coupled bosons on the four-dimensional Moyal plane, a particular example for a noncommutative geometry. The difference to the previous case is that now the external Yang–Mills fields act on wave functions by the noncommutative Moyal multiplication. This leads in a natural way to the generalized Klein–Gordon operator $\square_A^\theta$. As it turns out, the machinery of ΨDOs is still applicable, with (44) generalizing to

$$\sigma[f(\square_A^\theta)](x,p) = f(-p^2 + \square^\theta_{A(\cdot-(1/2)\Theta p)} + 2ip^\mu D^\theta_{A(\cdot-(1/2)\Theta p)\mu})1. \tag{45}$$

Here, $A(\cdot - \frac{1}{2}\Theta p)$ denotes the external fields A shifted by the amount $\frac{1}{2}\Theta p$.

From this formula, one might think that the new p-dependence in the external fields leads to an improvement in the decay properties of the symbol for large p. This point of view is however misleading when one wants to draw conclusions for the asymptotic expansion of the regularized trace: By a change of variables, the p-dependence in the external fields disappears under the spatial integral of the trace. This fact comes solely from the noncompactness of $\mathbb{R}^4$. It may be viewed as another manifestation of the UV/IR mixing. A similar effect can be seen for instance in the example of an infinitely smoothing operator on $\mathbb{R}$ that has a nonvanishing trace, see the end of the fourth section of the Appendix. Therefore, on noncompact manifolds (commutative or noncommutative), arguments linking the asymptotic expansion of the regularized trace of an operator to the expansion of its symbol must be taken with caution. For our case, we propose to use the asymptotic expansion in p of the shifted symbol

$$\sigma[f(\square_A^\theta)](x + \tfrac{1}{2}\Theta p, p)$$

instead. This proposal is justified rigorously by operator theoretic arguments which show that the difference between the original operator and a certain truncation of the asymptotic expansion of the above shifted symbol is trace-class. Hence, it does not contribute to the divergent part of the regularized trace and we can safely exchange the full symbol by its truncation. This argument can even be extended to the commutative case, thereby proving that the coefficient of the $\log \Lambda$ part of the regularized trace is indeed given by the (noncompact) Wodzicki residue. The latter observation now can be used to explain why the expression for $c_{\log}$ is a gauge invariant quantity: Since a gauge transformation conjugates the Klein–Gordon operator by some unitary operator, the fact that $c_{\log}$ is gauge invariant is equivalent to the vanishing of the Wodzicki residue on commutators.

To conclude, we have seen that the methods of ΨDOs are a powerful tool for the investigation of the case studied here, yet they need to be modified in the described way for the case of the noncommutative Moyal plane. It would be interesting to see what modifications are necessary to study the coupling of gravity to the bosons through a varying metric in $\square_A$. This is presently under investigation.

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APPENDIX: DETAILS OF THE COMPUTATIONS

Computation of $c_{\log}(A)$

We are using the following convention for the Klein–Gordon operator:

$$\begin{aligned}\Box_A = D_A^\mu D_{A\mu} = (\partial^\mu + ieA^\mu)(\partial_\mu + ieA_\mu) &= \partial^\mu\partial_\mu + ie\partial^\mu A_\mu + 2ieA_\mu\partial^\mu - e^2A^\mu A_\mu \\ &= \Box_0 + ie\partial^\mu A_\mu + 2ieA_\mu - e^2A^\mu A_\mu.\end{aligned} \tag{A1}$$

Recall the definition of the symbol $\sigma[a]$ of a pseudodifferential operator a:

$$(af)(x) = \int_{\mathbb{R}^4}\frac{d^4p}{(2\pi)^4}\int_{\mathbb{R}^4}d^4y\, e^{ip\cdot(x-y)}\sigma[a](p,x)f(y). \tag{A2}$$

In the computation we need the symbol of the resolvent of the Klein–Gordon operator, i.e., of the operator $(c_1I+c_2\Box_A)^{-1}$. To determine an asymptotic expansion for this symbol we start with the following expression:

$$\begin{aligned}(c_1I + c_2\Box_A)af(x) &= c_1I + c_2(\partial^\mu\partial_\mu + ie\partial^\mu A_\mu + 2ieA_\mu\partial^\mu - e^2A^\mu A_\mu) \\ &\quad\times\int_{\mathbb{R}^4}\frac{d^4p}{(2\pi)^4}\int_{\mathbb{R}^4}d^4y\, e^{ip\cdot(x-y)}\sigma[a](p,x)f(y) \\ &= \int_{\mathbb{R}^4}\frac{d^4p}{(2\pi)^4}\int_{\mathbb{R}^4}d^4y\, e^{ip\cdot(x-y)}(c_1I + c_2(-p^2 - 2eA_\mu p^\mu + \partial^\mu\partial_\mu \\ &\quad + ie\partial^\mu A_\mu + 2ieA_\mu\partial^\mu + 2ip_\mu\partial^\mu - e^2A^\mu A_\mu))\sigma[a](p,x)f(y).\end{aligned} \tag{A3}$$

Next replacing a by $(c_1I+c_2\Box_A)^{-1}\,a$ we get

$$\begin{aligned}(c_1I + c_2\Box_A)(c_1I + c_2\Box_A)^{-1}af(x) = af(x) &= \int_{\mathbb{R}^4}\frac{d^4p}{(2\pi)^4}\int_{\mathbb{R}^4}d^4y\, e^{ip\cdot(x-y)}(c_1I + c_2(-p^2 - 2eA_\mu p^\mu \\ &\quad + \partial^\mu\partial_\mu + ie\partial^\mu A_\mu + 2ieA_\mu\partial^\mu + 2ip_\mu\partial^\mu - e^2A^\mu A_\mu)) \\ &\quad\times\sigma[(c_1I + c_2\Box_A)^{-1}a](p,x)f(y) \\ &= \int_{\mathbb{R}^4}\frac{d^4p}{(2\pi)^4}\int_{\mathbb{R}^4}d^4y\, e^{ip\cdot(x-y)}\sigma[a](p,x)f(y).\end{aligned} \tag{A4}$$

So we have

$$\begin{aligned}&(c_1I + c_2(-p^2 - 2eA_\mu p^\mu + \partial^\mu\partial_\mu + ie\partial^\mu A_\mu + 2ieA_\mu\partial^\mu - e^2A^\mu A_\mu + 2ip_\mu\partial^\mu))\sigma[(c_1I + c_2\Box_A)^{-1}a](p,x) \\ &\quad = \sigma[a](p,x)\end{aligned} \tag{A5}$$

which can be written as

$$(c_1 I - c_2 p^2)\sigma[(c_1 I + c_2\Box_A)^{-1}a](p,x) + c_2(\Box_A + 2ip_\mu D_A^\mu)\sigma[(c_1 I + c_2\Box_A)^{-1}a](p,x) = \sigma[a](p,x), \tag{A6}$$

giving us the recursive relation

$$\sigma[(c_1 I + c_2\Box_A)^{-1}a](p,x) = \frac{1}{c_1 - c_2 p^2}\sigma[a](p,x) - \frac{c_2}{c_1 - c_2 p^2}(\Box_A + 2ip_\mu D_A^\mu)\sigma[(c_1 I + c_2\Box_A)^{-1}a](p,x). \tag{A7}$$

We can now get the desired asymptotic expansion by setting $a=1, \sigma[a]=1$,

$$\sigma[(c_1 I + c_2\Box_A)^{-1}](p,x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n c_2^n}{(c_1 - c_2 p^2)^{n+1}}(\Box_A + 2ip_\mu D_A^\mu)^n 1. \tag{A8}$$

Next we evaluate explicitly the terms contributing to the logarithmically diverging part in the expansion (25) of the effective action. When taking the angular integrals the following formulas are used:

$$\langle p^\mu p^\nu \rangle = \tfrac{1}{4}p^2 \eta^{\mu\nu},$$

$$\langle p^{\mu_1}p^{\mu_2}p^{\mu_3}p^{\mu_4}\rangle = \tfrac{1}{24}p^4(\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + \eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}),$$

where the brackets denote integration over the unit sphere in $\mathbb{R}^4$, that is

$$\langle f(p)\rangle := \frac{1}{2\pi^2}\int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4}\delta(|p|-1)f(p). \tag{A9}$$

Also the angular integral over an odd number of components p^μ is zero. The s-integrals in the expansion can be evaluated exactly using the formula

$$\int_0^1 ds \frac{s^{n-1}}{(1+sa)^{n+1}} = \frac{1}{n(1+a)^n} \tag{A10}$$

which holds for $\mathfrak{Re}\, a > 0$. The effective action can now be written as

$$\mathrm{Tr}_\Lambda\left(\log\left(\frac{-\Box_A + m^2}{\Lambda_0^2}\right) - \log\left(\frac{-\Box_0 + m^2}{\Lambda_0^2}\right)\right)$$
$$= -\frac{1}{(2\pi)^4}\sum_{n=1}^{\infty}\int_1^\Lambda d|p||p|^3 \frac{1}{n(p^2+m^2)^n}\int_{\mathbb{R}^4} d^4x \int_{S^3} d\Omega_p \mathrm{tr}_N(\Box_A + 2ip_\mu D_A^\mu)^n 1 + \text{const}, \tag{A11}$$

where the constant (in Λ) term arises from the integration of the symbol over the region $|p| \leq 1$ for which the asymptotic expansion is not valid.

When expanding the integrand in terms of p, the leading term is of the order p^{3-2n} times a term of order at most p^n coming from the angular integration—so the highest order term is of order p^{3-n}. For the terms contributing to logarithmic divergences of the effective action the leading order must be larger than or equal to -1, so the relevant terms in the expansion above are terms of order up to four. To find the parts contributing to the logarithmic divergence we derivate the terms in (A11) with respect to Λ and then pick the terms proportional to $1/\Lambda$. We denote by I_n the nth term in the expansion. Writing the expansions of the first four terms explicitly we then have

$$\frac{\partial I_1}{\partial \Lambda} = -\frac{1}{8\pi^2}\frac{\Lambda^3}{\Lambda_0^2}\frac{\Lambda_0^2}{\Lambda^2+m^2}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}_N \Box_A 1 = \frac{1}{8\pi^2}\left(\Lambda - m^2\frac{1}{\Lambda} + O\left(\frac{1}{\Lambda^3}\right)\right)\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}_N \Box_A 1,$$

$$\frac{\partial I_2}{\partial \Lambda} = -\frac{1}{8\pi^2}\frac{\Lambda^3}{2(\Lambda^2+m^2)^2}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}_N \Box_A^2 1 - \frac{1}{8\pi^2}\frac{\Lambda^5}{(\Lambda^2+m^2)^2}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}_N \Box_A 1$$
$$= -\frac{1}{8\pi^2}\frac{1}{2\Lambda}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}_N \Box_A^2 1 + \cdots + \frac{1}{8\pi^2}m^2\frac{1}{\Lambda}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}_N \Box_A 1 + \cdots ,$$

$$\frac{\partial I_3}{\partial \Lambda} = -\frac{1}{8\pi^2}\frac{\Lambda^3}{3(\Lambda^2+m^2)^3}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{Tr}_N \Box_A^3 + \frac{1}{4\pi^2}\frac{\Lambda^5}{3(\Lambda^2+m^2)^3}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}_N \Box_A^2$$
$$+ \frac{1}{8\pi^2}\frac{\Lambda^5}{3(\Lambda^2+m^2)^3}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}_N (D_A^\mu \Box_A D_{A\mu})$$
$$= -\frac{1}{8\pi^2}O\left(\frac{1}{\Lambda^3}\right) + \cdots + \frac{1}{4\pi^2}\frac{1}{3\Lambda}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}_N \Box_A^2 + \cdots + \frac{1}{8\pi^2}\frac{1}{3\Lambda}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}_N (D_A^\mu \Box_A D_{A\mu}),$$

$$\frac{\partial I_4}{\partial \Lambda} = -\frac{1}{8\pi^2}\frac{\Lambda^3}{4(\Lambda^2+m^2)^4}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}_N\Big(\Box_A^4 - \Lambda^2(3\Box_A^3 + \Box_A D_A^\mu \Box_A D_{A\mu} + D_{A\mu}\Box_A^2 D_{A\mu}) + \frac{2}{3}\Lambda^4(\Box_A^2$$
$$+ D_A^\nu D_A^\mu D_{A\nu} D_{A\mu} + D_A^\mu \Box_A D_{A\mu})\Big)$$
$$= -\frac{1}{8\pi^2}\frac{2}{3}\frac{1}{4\Lambda}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}_N(\Box_A^2 + D_A^\nu D_A^\mu D_{A\nu} D_{A\mu} + D_A^\mu \Box_A D_{A\mu}) + \cdots .$$

Comparison with fermion calculations

We now compute the traces of the relevant terms in the identity

$$\log(-\mathbb{1}_4(\Box_A + m^2) - ie\sigma\cdot F) = \log\left(\frac{\mathbb{1}_4(-\Box_A + m^2)}{\Lambda_0^2}\right) + \log\left(\mathbb{1}_4 - \frac{ie}{-\Box_A + m^2}\sigma\cdot F\right)$$
$$+ \frac{1}{2}\left[\log\left(\mathbb{1}_4\frac{-\Box_A + m^2}{\Lambda_0^2}\right), \log\left(\mathbb{1}_4 - \frac{ie}{-\Box_A + m^2}\sigma\cdot F\right)\right]$$
$$+ \frac{1}{12}\left(\left[\log\left(\mathbb{1}_4\frac{-\Box_A + m^2}{\Lambda_0^2}\right), \left[\log\left(\mathbb{1}_4\frac{-\Box_A + m^2}{\Lambda_0^2}\right),\right.\right.\right.$$
$$\left.\left.\log\left(\mathbb{1}_4 - \frac{ie}{-\Box_A + m^2}\sigma\cdot F\right)\right]\right] + \left[\log\left(\mathbb{1}_4 - \frac{ie}{-\Box_A + m^2}\sigma\cdot F\right),\right.$$
$$\left.\left.\left[\log\left(\mathbb{1}_4 - \frac{ie}{-\Box_A + m^2}\sigma\cdot F\right), \log\left(\mathbb{1}_4\frac{-\Box_A + m^2}{\Lambda_0^2}\right)\right]\right]\right) + O\left(\frac{1}{\Lambda^5}\right).$$

The commutator terms come from the Baker–Campbell–Hausdorff formula. Terms that fall off at least as $1/\Lambda^5$ have been suppressed. We find

$$\mathrm{Tr}_\Lambda \log\left(\mathbb{1}_4\left(\frac{-\Box_A+m^2}{\Lambda_0^2}\right)\right) = 4\,\mathrm{Tr}_\Lambda \log\left(\frac{-\Box_A+m^2}{\Lambda_0^2}\right),$$

$$\mathrm{Tr}_\Lambda \log\left(\mathbb{1}_4 - \frac{ie}{m^2-\Box_A}\sigma\cdot F\right) = -\int_{|p|\leqslant\Lambda}\frac{\mathrm{d}^4p}{(2\pi)^4}\int_{\mathbb{R}^4}\mathrm{d}^4x\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}\left(\sigma\left[\frac{ie}{-\Box_A+m^2}\right]*\sigma\cdot F\right)^{*n}.$$

Here, $*$ denotes the product of symbols of two ΨDOs which has the asymptotic expansion (20).

Now using the fact that $\mathrm{tr}\,\sigma\cdot F=0$ and the expansion (A8) for $\sigma[1/(c_1+c_2\Box_A)]$, we get

$$\begin{aligned}
&\mathrm{Tr}_\Lambda \log\left(\mathbb{1}_4 - \frac{ie}{m^2-\Box_A}\sigma\cdot F\right)\\
&\quad= -\frac{1}{2}\int_{|p|\leqslant\Lambda}\frac{\mathrm{d}^4p}{(2\pi)^4}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}\left(\sigma\left[\frac{ie}{-\Box_A+m^2}\right]*\sigma\cdot F*\sigma\left[\frac{ie}{-\Box_A+m^2}\right]*\sigma\cdot F\right)+O\left(\frac{1}{\Lambda^5}\right)\\
&\quad= \frac{1}{2}\int_{|p|\leqslant\Lambda}\frac{\mathrm{d}^4p}{(2\pi)^4}\int_{\mathbb{R}^4}\mathrm{d}^4x\frac{e^2}{(p^2+m^2)^2}\,\mathrm{tr}\,\sigma\cdot F^2+O\left(\frac{1}{\Lambda^5}\right)\\
&\quad= \frac{e^2}{16\pi^2}\log\Lambda\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}\,\sigma\cdot F^2+O(\Lambda^0).
\end{aligned}$$

This provides the results needed in the main text, since, as will be shown below, there are no contributions to the divergent part of the trace that come from the commutator terms. For this, we expand the logarithm in the first commutator term above which gives us

$$\begin{aligned}
&\frac{1}{2}\mathrm{Tr}_\Lambda\left[\log\left(\mathbb{1}_4\frac{-\Box_A+m^2}{\Lambda_0^2}\right),\log\left(\mathbb{1}_4-\frac{ie}{-\Box_A+m^2}\sigma\cdot F\right)\right]\\
&\quad=\frac{1}{2}\mathrm{Tr}_\Lambda\left[\log\left(\mathbb{1}_4\frac{-\Box_A+m^2}{\Lambda_0^2}\right),\frac{-ie}{-\Box_A+m^2}\sigma\cdot F\right]+\frac{1}{2}\mathrm{Tr}_\Lambda\left[\log\left(\mathbb{1}_4\frac{(-\Box_A+m^2)}{\Lambda_0^2}\right),\right.\\
&\qquad\left.-\frac{1}{2}\left(\frac{ie}{-\Box_A+m^2}\sigma\cdot F\right)^2\right]+O\left(\frac{1}{\Lambda^7}\right).
\end{aligned}$$

The first term on the rhs is zero, so we have

$$\begin{aligned}
&\frac{1}{2}\mathrm{Tr}_\Lambda\left[\log\left(\frac{\mathbb{1}_4(-\Box_A+m^2)}{\Lambda_0^2}\right),\log\left(\mathbb{1}_4-\frac{ie}{-\Box_A+m^2}\sigma\cdot F\right)\right]\\
&\quad=\frac{1}{4}e^2\,\mathrm{Tr}_\Lambda\left[\log\left(\frac{\mathbb{1}_4(-\Box_A+m^2)}{\Lambda_0^2}\right),\left(\frac{1}{-\Box_A+m^2}\sigma\cdot F\right)^2\right]+O\left(\frac{1}{\Lambda^7}\right).
\end{aligned}$$

Now

$$\begin{aligned}
\sigma\left[\left[\log\left(\mathbb{1}_4\frac{(-\Box_A+m^2)}{\Lambda_0^2}\right),\left(\frac{1}{-\Box_A+m^2}\sigma\cdot F\right)^2\right]\right]&=\left[\log\frac{p^2+m^2}{\Lambda_0^2},\left(\frac{1}{p^2+m^2}\right)^2\sigma\cdot F^2\right]_*+O\left(\frac{1}{\Lambda^6}\right)\\
&=-i\frac{2p^\mu\Lambda_0^2}{p^2+m^2}\left(\frac{1}{p^2+m^2}\right)^2\partial_\mu(\sigma\cdot F^2)+O\left(\frac{1}{\Lambda^6}\right)\\
&=O\left(\frac{1}{\Lambda^5}\right)
\end{aligned}$$

so there is no contribution to the divergent part of the trace.

Next, we turn to the first triple commutator term in the above identity. Counting the powers of p in the pertinent symbols, the leading term should scale like $1/p^4$. However, this term contains a single $\sigma\cdot F$ which gives zero under the trace tr_N. Therefore, one must take one more term in the expansion of

$$\log\left(\mathbb{1}_4 - \frac{ie}{-\Box_A + m^2}\sigma\cdot F\right).$$

The resulting expression then is of order $1/p^6$ and hence can be dropped. Finally the second triple commutator term can be seen to behave as $1/p^6$. In conclusion, we have shown that for the divergent terms of the cutoff regularized trace, all commutator terms can be neglected in the above identity.

Dependence on the regularization scheme

Computation of the symbol of $\theta(\Lambda+\Box_A)$. We start with the following sum expression for the (regularized) θ-function (Ref. 9, p. 248, etc.):

$$\theta_\epsilon(x) = \frac{1}{\epsilon}\sum_{r=-\infty}^{\infty}\frac{e^{i\omega_r 0^+}}{x+i\omega_r} = \frac{e^{-x0^+}}{e^{-x\epsilon}+1}, \quad \omega_r = (2r+1)\pi/\epsilon, \quad \epsilon > 0.$$

(This expression is the discretized version of the well-known integral formula

$$\theta(x) = \int_{\mathbb{R}}\frac{\mathrm{d}z}{2\pi i}\frac{e^{iz0^+}}{z-ix}.$$

The latter is regained for $\epsilon\to\infty$.) Differentiation yields

$$\delta_\epsilon^{(n-1)}(x) = \frac{1}{\epsilon}\sum_{r=-\infty}^{\infty}\frac{e^{i\omega_r 0^+}(-1)^n n!}{(x+i\omega_r)^{n+1}}, \quad n = 1,2,3,\ \dots,$$

for the $(n-1)$th derivative of the (regularized) Dirac δ-function.

Using the above expression, we have

$$\sigma[\theta_\epsilon(\Lambda^2+\Box_A)] = \frac{1}{\epsilon}\sum_{r=-\infty}^{\infty}\sigma\left[\frac{e^{i\omega_r 0^+}}{(\Lambda^2+\Box_A)+i\omega_r}\right].$$

We derived the asymptotic expansion for the symbol of $(c_1 I + c_2\Box_A)^{-1}$ [see Eq. (A8)] to be given by

$$\sigma[(c_1 I + c_2\Box_A)^{-1}](p,x) \sim \sum_{n=0}^{\infty}\frac{(-1)^n c_2^n}{(c_1 - c_2 p^2)^{n+1}}(\Box_A + 2ip_\mu D_A^\mu)^n 1.$$

Using this we get

$$\sigma[\theta_\epsilon(\Lambda+\Box_A)] \sim \frac{1}{\epsilon}\sum_{r=-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{e^{i\omega_r 0^+}(-1)^n}{((\Lambda^2-p^2)+i\omega_r)^{n+1}}(\Box_A + 2ip^\mu D_{A\mu})^n 1.$$

Using the expressions for $\delta_\epsilon^{(n)}$ in the above expansion we finally have

$$\sigma[\theta_\epsilon(\Lambda+\Box_A)] \sim \sum_{n=0}^{\infty}\frac{1}{n!}\delta_\epsilon^{(n-1)}(\Lambda^2-p^2)(\Box_A + 2ip^\mu D_{A\mu})^n 1.$$

Computation of the traces: We can now proceed with the calculation of the trace. From the remarks in the main section, we know that we can drop the spatial regulator f since terms proportional to the volume of $\mathbb{R}^4$ cancel exactly. We calculate

$$\mathrm{Tr}_\Lambda^{\Box_A} \log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right) - \mathrm{Tr}_\Lambda \log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\int_{|p|\geq 1} \frac{\mathrm{d}^4p}{(2\pi)^4}\int_{\mathbb{R}^4} \mathrm{d}^4x\ \delta_\epsilon^{(n-1)}(\Lambda^2-p^2)\log\left(\frac{p^2+m^2}{\Lambda_0^2}\right)\mathrm{tr}_N(\Box_A+2ip^\mu D_{A\mu})^n 1 + \cdots$$

$$- \mathrm{Tr}_\Lambda \log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right)$$

$$= \int_{|p|\geq 1} \frac{\mathrm{d}^4p}{(2\pi)^4}\int_{R^4} \mathrm{d}^4x\ \theta_\epsilon(\Lambda^2-p^2)\log\left(\frac{p^2+m^2}{\Lambda_0^2}\right)\mathrm{tr}_N\, 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\int_{|p|\geq 1}\frac{\mathrm{d}^4p}{(2\pi)^4}\int_{\mathbb{R}^4}\delta_\varepsilon^{(n-1)}(\Lambda^2-p^2)$$

$$\times\log\left(\frac{p^2+m^2}{\Lambda_0^2}\right)\mathrm{tr}_N(\Box_A+2ip^\mu D_{A\mu})^n 1 + \cdots - \mathrm{Tr}_\Lambda \log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right),$$

where the dots indicate terms that are uniformly bounded in Λ. (In particular, we have split the p-integral in a part over the unit ball and an integral over the rest. The former contributes to the finite part.) Now the first term on the right-hand side matches the last one in the limit $\epsilon\to\infty$. As explained in the main text, we are interested in the terms $n\leq 5$ of the sum above. Expanding the pertinent terms and performing the angular p-integrals gives

$$\mathrm{Tr}_\Lambda^{\Box_A} \log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right) - \mathrm{Tr}_\Lambda \log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right)$$

$$= \frac{1}{8\pi^2}\int_1^\infty \mathrm{d}p\ p^3\delta(\Lambda^2-p^2)\log\left(\frac{p^2+m^2}{\Lambda_0^2}\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\ \mathrm{tr}_N\,\Box_A 1 + \frac{1}{2}\frac{1}{8\pi^2}\int_1^\infty \mathrm{d}p\ p^3\delta^{(1)}(\Lambda^2-p^2)$$

$$\times\log\left(\frac{p^2+m^2}{\Lambda_0^2}\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\ \mathrm{tr}_N(\Box_A^2-p^2\Box_A)1 + \frac{1}{6}\frac{1}{8\pi^2}\int_1^\infty \mathrm{d}p\ p^3\delta^{(2)}(\Lambda^2-p^2)$$

$$\times\log\left(\frac{p^2+m^2}{\Lambda_0^2}\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\ \mathrm{tr}_N(\Box_A^3-p^2(2\Box_A^2+D_{A\mu}\Box_A D_{A\mu}))1$$

$$+\frac{1}{24}\frac{1}{8\pi^2}\int_1^\infty \mathrm{d}p\ p^3\delta^{(3)}(\Lambda^2-p^2)\log\left(\frac{p^2+m^2}{\Lambda_0^2}\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\ \mathrm{tr}_N\Big(\Box_A^4-3p^2\Box_A^3$$

$$-p^2\Box_A D_{A\mu}\Box_A D_A^\mu - p^2 D_{A\mu}\Box_A^2 D_{A\mu} - p^2 D_{A\mu}\Box_A D_A^\mu\Box_A + \tfrac{2}{3}p^4(\Box_A^2$$

$$+D_A^\mu D_A^\nu D_{A\mu}D_{A\nu} + D_A^\mu\Box_A D_{A\mu})\Big)1 + \cdots .$$

We have also taken the limit $\epsilon\to\infty$, in which δ_ϵ goes over into the Dirac δ-function.

Gathering terms with equal spatial integral we obtain

$$
\begin{aligned}
&\mathrm{Tr}_{\Lambda}^{\Box_A}\log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right)-\mathrm{Tr}_{\Lambda}^{\Box_0}\log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right)\\
&\quad=\frac{1}{8\pi^2}\int_1^\infty \mathrm{d}p\,\log\left(\frac{p^2+m^2}{\Lambda_0^2}\right)\left(p^3\delta(\Lambda^2-p^2)-\frac{1}{2}p^5\delta^{(1)}(\Lambda^2-p^2)\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,\Box_A 1\\
&\qquad+\frac{1}{8\pi^2}\int_1^\infty \mathrm{d}p\,\log\left(\frac{p^2+m^2}{\Lambda_0^2}\right)\left(\frac{1}{2}p^3\delta^{(1)}(\Lambda^2-p^2)-\frac{1}{3}p^5\delta^{(2)}(\Lambda^2-p^2)\right.\\
&\qquad\left.+\frac{1}{36}p^7\delta^{(3)}(\Lambda^2-p^2)\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,\Box_A^2 1+\frac{1}{8\pi^2}\int_1^\infty \mathrm{d}p\,\log\left(\frac{p^2+m^2}{\Lambda_0^2}\right)\\
&\qquad\times\left(-\frac{1}{6}p^5\delta^{(2)}(\Lambda^2-p^2)+\frac{1}{36}p^7\delta^{(3)}(\Lambda^2-p^2)\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,D_A^\mu\Box_A D_{A\mu}1\\
&\qquad+\frac{1}{8\pi^2}\int_1^\infty \mathrm{d}p\,\log\left(\frac{p^2+m^2}{\Lambda_0^2}\right)\frac{1}{36}p^7\delta^{(3)}(\Lambda^2-p^2)\\
&\qquad\times\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,D_A^\mu D_A^\nu D_{A\mu}D_{A\nu}1+\frac{1}{8\pi^2}\int_1^\infty dp\,\log\left(\frac{p^2+m^2}{\Lambda_0^2}\right)\left(\frac{1}{6}p^3\delta^{(3)}(\Lambda^2-p^2)\right.\\
&\qquad\left.-\frac{1}{8}p^5\delta^{(4)}(\Lambda^2-p^2)\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,\Box_A^3 1+\cdots\,.
\end{aligned}
$$

Next we make a change of variables $p^2=u$ to get

$$
\begin{aligned}
&\mathrm{Tr}_{\Lambda}^{\Box_A}\log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right)-\mathrm{Tr}_{\Lambda}^{\Box_0}\log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right)\\
&\quad=\frac{1}{8\pi^2}\int_1^\infty \frac{\mathrm{d}u}{2}\log\left(\frac{u+m^2}{\Lambda_0^2}\right)\left(u\delta(\Lambda^2-u)-\frac{1}{2}u^2\delta^{(1)}(\Lambda^2-u)\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,\Box_A 1\\
&\qquad+\frac{1}{8\pi^2}\int_1^\infty \frac{\mathrm{d}u}{2}\log\left(\frac{u+m^2}{\Lambda_0^2}\right)\left(\frac{1}{2}u\delta^{(1)}(\Lambda^2-u)-\frac{1}{3}u^2\delta^{(2)}(\Lambda^2-u)\right.\\
&\qquad\left.+\frac{1}{36}u^3\delta^{(3)}(\Lambda^2-u)\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,\Box_A^2 1+\frac{1}{8\pi^2}\int_1^\infty \frac{\mathrm{d}u}{2}\log\left(\frac{u+m^2}{\Lambda_0^2}\right)\\
&\qquad\times\left(-\frac{1}{6}u^2\delta^{(2)}(\Lambda^2-u)+\frac{1}{36}u^3\delta^{(3)}(\Lambda^2-u)\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,D_A^\mu\Box_A D_{A\mu}1\\
&\qquad+\frac{1}{8\pi^2}\int_1^\infty \frac{\mathrm{d}u}{2}\log\left(\frac{u+m^2}{\Lambda_0^2}\right)\frac{1}{36}u^3\delta^{(3)}(\Lambda^2-u)\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,D_A^\mu D_A^\nu D_{A\mu}D_{A\nu}1\\
&\qquad+\frac{1}{8\pi^2}\int_0^\infty \frac{\mathrm{d}u}{2}\log\left(\frac{u+m^2}{\Lambda_0^2}\right)\left(\frac{1}{6}u\delta^{(2)}(\Lambda^2-u)-\frac{1}{8}u^2\delta^{(3)}(\Lambda^2-u)\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}_N\,\Box_A^3 1+\cdots\,.
\end{aligned}
$$

Now by integrating by parts and noting that

$$\frac{\mathrm{d}^k}{\mathrm{d}u^k}\delta(\Lambda^2-u)=(-1)^k\delta^{(k)}(\Lambda^2-u)$$

we have

$$
\begin{aligned}
&\mathrm{Tr}_\Lambda^{\Box_A} \log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right) - \mathrm{Tr}_\Lambda^{\Box_0} \log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right) \\
&\quad = \frac{1}{16\pi^2}\int_1^\infty \mathrm{d}u \log\left(\frac{u+m^2}{\Lambda_0^2}\right)\delta(\Lambda^2-u)u(1-1)\int_{\mathbb{R}^4}\mathrm{d}^4x\, \mathrm{tr}_N\, \Box_A 1 \\
&\qquad + \frac{1}{16\pi^2}\int_1^\infty \mathrm{d}u \log\left(\frac{u+m^2}{\Lambda_0^2}\right)\delta(\Lambda^2-u)\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{6}\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\, \mathrm{tr}_N\, \Box_A^2 1 \\
&\qquad + \frac{1}{16\pi^2}\int_1^\infty du \log\left(\frac{u+m^2}{\Lambda_0^2}\right)\delta(\Lambda^2-u)\left(-\frac{1}{3}+\frac{1}{6}\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\, \mathrm{tr}_N\, D_A^\mu \Box_A D_{A\mu} 1 \\
&\qquad + \frac{1}{16\pi^2}\int_1^\infty \mathrm{d}u \log\left(\frac{u+m^2}{\Lambda_0^2}\right)\frac{1}{6}\delta(\Lambda^2-u)\int_{\mathbb{R}^4}\mathrm{d}^4x\, \mathrm{tr}_N\, D_A^\mu D_A^\nu D_{A\mu} D_{A\nu} 1 + \cdots \\
&\quad = -\frac{1}{16\pi^2}\frac{1}{6}\int_1^\infty \mathrm{d}u \log\left(\frac{u+m^2}{\Lambda_0^2}\right)\delta(\Lambda^2-u)\int_{\mathbb{R}^4}\mathrm{d}^4x\, \mathrm{Tr}\, D_A^\mu \Box_A D_{A\mu} 1 + \frac{1}{16\pi^2}\frac{1}{6} \\
&\qquad \times \int_1^\infty \mathrm{d}u \log\left(\frac{u+m^2}{\Lambda_0^2}\right)\delta(\Lambda^2-u)\int_{\mathbb{R}^4}\mathrm{d}^4x\, \mathrm{tr}_N\, D_A^\mu D_A^\nu D_{A\mu} D_{A\nu} 1 + \cdots \\
&\quad = -\frac{1}{96\pi^2}\log\left(\frac{\Lambda^2+m^2}{\Lambda_0^2}\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\, \mathrm{tr}_N\, D_A^\mu \Box_A D_{A\mu} 1 \\
&\qquad + \frac{1}{96\pi^2}\log\left(\frac{\Lambda^2+m^2}{\Lambda_0^2}\right)\int_{\mathbb{R}^4}\mathrm{d}^4x\, \mathrm{tr}_N\, D_A^\mu D_A^\nu D_{A\mu} D_{A\nu} 1 + \cdots\ .
\end{aligned}
$$

Recalling that

$$
\mathrm{tr}_N(D_A^\mu \Box_A D_{A\mu} - D_A^\nu D_A^\mu D_{A\nu} D_{A\mu}) = \frac{e^2}{2}\,\mathrm{tr}_N\, F^{\mu\nu}F_{\mu\nu}
$$

we finally get

$$
\mathrm{Tr}_\Lambda^{\Box_A} \log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right) - \mathrm{Tr}_\Lambda^{\Box_0} \log\left(\frac{-\Box_0+m^2}{\Lambda_0^2}\right) = -\frac{1}{2}\frac{1}{96\pi^2}\log\frac{\Lambda}{\Lambda_0}\int_{\mathbb{R}^4}\mathrm{tr}_N\, F^{\mu\nu}F_{\mu\nu} + \cdots\ , \tag{A12}
$$

where again the dots indicate terms that are bounded or polynomial in Λ.

Computation on the Moyal plane

General remarks: The symbol of the operator $c_1+c_2\Box_A^\theta$ is given by

$$
\begin{aligned}
\sigma(x,p) &:= \sigma[c_1+c_2\Box_A^\theta](x,p) \\
&= -p^2 - 2ep^\mu A_\mu(x-\tfrac{1}{2}\Theta p) + ie(\partial^\mu A_\mu)(x-\tfrac{1}{2}\Theta p) - e^2(A^\mu\star A_\mu)(x-\tfrac{1}{2}\Theta p).
\end{aligned}
$$

From this expression, it is clear that one can bound σ from below by a positive constant and from above by a multiple of p^2 for p^2 greater than a certain constant. Furthermore, the derivatives of σ fall off as long as x is confined to some compact set. Therefore, by Ref. 17, corollary 5.1, there is a ΨDO that inverts $(c_1+c_2\Box_A^\theta)$ up to some infinitely smoothing operator.

Derivation of the recursion relation: As in the first section of this appendix, we start with the following identity for σ:

$$\begin{aligned}
\psi(x) &= (c_1 + c_2\Box_A^\theta)(c_1 + c_2\Box_A^\theta)^{-1}\psi(x) \\
&= (c_1 + c_2\Box_A^\theta)\int \frac{\mathrm{d}^4p}{(2\pi)^4}\int \mathrm{d}^4y\, e^{ip(x-y)}\sigma(x,p)\psi(y) \\
&= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\int \mathrm{d}^4y(c_1 + c_2(\partial^\mu\partial_\mu + ie(\partial^\mu A_\mu)\star + 2ieA^\mu\star\partial_\mu - e^2A^\mu\star A_\mu\star))(e^{ip(x-y)}\sigma[(c_1 \\
&\quad + c_2\Box_A^\theta)^{-1}](x,p))\psi(y).
\end{aligned}$$

To continue we need the following formula:

$$e^{ip(x-y)}\sigma(x,p) = [\sigma(\cdot + \tfrac{1}{2}\Theta p,p)\star e^{ip(\cdot-y)}\chi(\cdot)](x),$$

which can be proved as follows. Using the integral expression for the star product,

$$(f\star g)(x) := (2\pi)^{-4}\int\int e^{i\xi(x-y)}f\left(x - \frac{1}{2}\Theta\xi\right)g(y)\mathrm{d}^4y\, \mathrm{d}^4\xi,$$

we have for a Schwartz test function χ

$$\begin{aligned}
\left[\sigma\left(\cdot + \frac{1}{2}\Theta p,p\right)\star e^{ip(\cdot-y)}\chi(\cdot)\right](x) &= \frac{1}{(2\pi)^4}\int\int \mathrm{d}^4\xi\, \mathrm{d}^4z\sigma\left(x - \frac{1}{2}\theta\xi + \frac{1}{2}\Theta p,p\right)e^{ip(z-y)}\chi(z)e^{i\xi(x-z)} \\
&= \frac{1}{(2\pi)^4}\int\int \mathrm{d}^4\xi\, \mathrm{d}^4z\sigma\left(x - \frac{1}{2}\theta\xi + \frac{1}{2}\Theta p,p\right)\chi(z)e^{-iz(\xi-p)}e^{i(\xi x-py)} \\
&= \frac{1}{(2\pi)^2}\int \mathrm{d}^4\xi\, \sigma\left(x - \frac{1}{2}\Theta(\xi - p),p\right)\hat{\chi}(\xi - p)e^{i(\xi x-py)} \\
&= e^{ip(x-y)}\frac{1}{(2\pi)^2}\int \mathrm{d}^4\tilde{\xi}\, \sigma\left(x - \frac{1}{2}\Theta\tilde{\xi},p\right)\hat{\chi}(\tilde{\xi})e^{i\tilde{\xi}x}.
\end{aligned}$$

Now in the limit $\chi \to 1$, the Fourier transform $\hat{\chi}$ approximates the delta function. Therefore, in this limit, we obtain the claimed identity. Using this formula in the expression for $\psi(x)$ we get

$$\begin{aligned}
\psi(x) &= \int\int \frac{\mathrm{d}^4p\, \mathrm{d}^4y}{(2\pi)^4}(c_1 + c_2(\partial^\mu\partial_\mu + ie(\partial^\mu A_\mu)\star + 2ieA^\mu\star\partial_\mu - e^2A^\mu\star A_\mu\star))(\sigma(x,p)e^{ip(x-y)})\psi(y) \\
&= \int\int \frac{\mathrm{d}^4p\, \mathrm{d}^4y}{(2\pi)^4}(c_1 + c_2(\partial^\mu\partial_\mu + ie(\partial^\mu A_\mu)\star + 2ieA^\mu\star\partial_\mu - e^2A^\mu\star A_\mu\star)) \\
&\quad \times\left(\sigma\left(\cdot + \frac{1}{2}\Theta p,p\right)\star e^{ip(\cdot-y)}\right)(x)\psi(y) \\
&= \int\int \frac{\mathrm{d}^4p\, \mathrm{d}^4y}{(2\pi)^4}\Bigg[(c_1 + c_2(\Box_0 + 2ip^\mu\partial_\mu - p^2 + ie(\partial^\mu A_\mu)\star - 2ep^\mu A_\mu\star \\
&\quad + 2ieA^\mu\star\partial_\mu - e^2A^\mu\star A_\mu\star))\sigma\left(\cdot + \frac{1}{2}\Theta p,p\right)\Bigg]\star e^{ip(\cdot-y)}(x)\psi(y) \\
&= \int\int \frac{\mathrm{d}^4p\, \mathrm{d}^4y}{(2\pi)^4}(c_1 + c_2p^2 + c_2(\Box^\theta_{A(\cdot-(1/2)\Theta p)} + 2p^\mu D^\theta_{A(\cdot-(1/2)\Theta p),\mu}))\sigma(\cdot,p)e^{ip(x-y)}\psi(y),
\end{aligned}$$

which gives us

$$1 = (c_1 - c_2p^2 + c_2(\Box^\theta_{A(\cdot-(1/2)\Theta p)} + 2ip^\mu D^\theta_{A(\cdot-(1/2)\Theta p),\mu}))\sigma[(c_1 + c_2\Box_A^\theta)^{-1}](x,p)$$

or

$$\sigma[(c_1+c_2\Box_A^\theta)^{-1}](x,p)=\frac{1}{c_1-c_2p^2}-\frac{c_2}{c_1-c_2p^2}(\Box^\theta_{A(\cdot-(1/2)\Theta p)}+2ip^\mu D^\theta_{A(\cdot-(1/2)\Theta p)})\sigma[(c_1+c_2\Box_A^\theta)^{-1}]\times(x,p). \tag{A13}$$

Derivation of the asymptotic expansion: We set $R:=(c_1+c_2\Box_A^\theta)^{-1}$. As $-\Box_A^\theta$ is a positive operator, R is bounded for $c_1\cdot c_2<0$. Indeed, from

$$\int_{\mathbb{R}^4}\mathrm{d}^4x\,\overline{\psi}(x)(A\star\varphi)(x)=\int_{\mathbb{R}^4}\mathrm{d}^4x(\overline{\psi}\star A\star\varphi)(x)=\int_{\mathbb{R}^4}\mathrm{d}^4x(\overline{\psi}\star A)(x)\varphi(x),$$

which holds for $\psi,A,\varphi\in L^2(\mathbb{R}^4)$ (Ref. 10, lemma 2.10) and $\overline{\overline{A}\star\psi}=\overline{\psi}\star A$ we conclude

$$\langle\psi,A\star\varphi\rangle=\langle\overline{A}\star\psi,\varphi\rangle$$

and hence $(D^\theta_{A\mu})^\dagger=-D^\theta_{A\mu}$. Therefore,

$$\langle\varphi,-\Box_A^\theta\varphi\rangle=\sum_{\mu=1}^4\langle D^\theta_{A\mu}\varphi,D^\theta_{A\mu}\varphi\rangle\geqslant 0.$$

In our case, we have $c_1=1-s+s(m^2/\Lambda_0^2)$ and $c_2=-s/\Lambda_0^2$ for $0\leqslant s\leqslant 1$ which meets the above requirement of $c_1\cdot c_2<0$ for $0<s\leqslant 1$. For $s=0$, we have $c_2=0$, $c_1\neq 0$, and R is a multiple of the identity.

Next, let R_N be the ΨDO defined by the symbol

$$\sigma[R_N](x,p)=\sum_{n=0}^N\frac{(-c_2)^n}{(c_1-c_2p^2)^{n+1}}(\Box^\theta_{A(\cdot-(1/2)\Theta p)}+2ip^\mu D^\theta_{A(\cdot-(1/2)\Theta p),\mu})^n 1.$$

We will show that the difference $R-R_N$ is a trace-class operator.

For this, we first apply $c_1+c_2\Box_A^\theta$ from the left to obtain

$$(c_1+c_2\Box_A^\theta)(R-R_N)=1-(c_1+c_2\Box_A^\theta)R_N.$$

Here, 1 denotes the identity operator. We will compute the symbol of the ΨDO on the right-hand side of this equation. On the level of symbols, multiplication of R_N by $c_1+c_2\Box_A^\theta$ from the left amounts to the application of $c_1+c_2(-p^2+\Box^\theta_{A(\cdot-(1/2)\Theta p)}+2ip^\mu D^\theta_{A(\cdot-(1/2)\Theta p),\mu})$ to $\sigma[R_N]$, cf. the derivation of the recursion relation above. Hence, we find

$$\begin{aligned}\sigma[1-(c_1+c_2\Box_A^\theta)R_N](x,p)&=1-(c_1+c_2(-p^2+\Box^\theta_{A(\cdot-(1/2)\Theta p)}+2ip^\mu D^\theta_{A(\cdot-(1/2)\Theta p),\mu}))\sigma[R_N](x,p)\\&=1-(c_1-c_2p^2)\sigma[R_N]-c_2(\Box^\theta_{A(\cdot-(1/2)\Theta p)}+2ip^\mu D^\theta_{A(\cdot-(1/2)\Theta p),\mu})\sigma[R_N]\\&=-\sum_{n=1}^N\frac{(-c_2)^n}{(c_1-c_2p^2)^n}(\Box^\theta_{A(\cdot-(1/2)\Theta p)}+2ip^\mu D^\theta_{A(\cdot-(1/2)\Theta p),\mu})^n 1\\&\quad+\sum_{n=0}^N\frac{(-c_2)^{n+1}}{(c_1-c_2p^2)^{n+1}}(\Box^\theta_{A(\cdot-(1/2)\Theta p)}+2ip^\mu D^\theta_{A(\cdot-(1/2)\Theta p),\mu})^{n+1}1\\&=\frac{(-c_2)^{N+1}}{(c_1-c_2p^2)^{N+1}}(\Box^\theta_{A(\cdot-(1/2)\Theta p)}+2ip^\mu D^\theta_{A(\cdot-(1/2)\Theta p),\mu})^{N+1}1.\end{aligned}$$

Let r_N be defined by the last expression,

$$\sigma[r_N](x,p) := \frac{(-c_2)^{N+1}}{(c_1 - c_2 p^2)^{N+1}} (\Box^{\theta}_{A(\cdot-(1/2)\Theta p)} + 2ip^{\mu} D^{\theta}_{A(\cdot-(1/2)\Theta p),\mu})^{N+1} 1.$$

We will show that r_N is a trace-class operator for sufficiently large N. Expanding the power of operators in the symbol $\sigma[r_N]$ yields terms of the form

$$\text{const} \times \frac{1}{(c_1 - c_2 p^2)^{N+1}} \times f_1 \star \cdots \star f_k \left(x - \frac{1}{2}\Theta p \right),$$

$k=1,\ldots,2(N+1)$, the f_i denoting the external fields A_μ or derivatives thereof. [We have used the fact that $(f(\cdot-\frac{1}{2}\Theta p) \star g(\cdot-\frac{1}{2}\Theta p))(x) = (f\star g)(x-\frac{1}{2}\Theta p)$.]

As A_μ is in $\mathcal{P}$ and of order $-2-\epsilon$, Moyal multiplication by it increases the decay property of the x-dependent part by 2. On the other hand, differentiation increases it only by 1. Therefore, the leading term of the above type will be the one where N derivatives of $D^{\theta}_{A(\cdot-(1/2)\Theta p),\mu}$ hit a single A_μ. The resulting term can be bounded from above by

$$\text{const} \times \frac{1}{(1+p^2)^{N+1}} \times (p^2)^{N/2} \times \frac{1}{(1 + (x - \frac{1}{2}\Theta p)^2)^{(4+\epsilon+N)/2}}$$

which is integrable in $x-p$ space for sufficiently large N.

Application from the left of the bounded operator R to r_N does not change the property of being trace-class. On the other hand, we find

$$R r_N = R(c_1 + c_2 \Box^{\theta}_{A})(R - R_N) = R - R_N.$$

To summarize, if we are interested in the singular behavior of the cutoff regularized trace of R, we may use the symbol $\sigma[R_N]$ for N sufficiently large in the integral formula of the trace. This amounts to the iteration of the recursion relation (A13) N times.

Remarks: It is easy to see that a blind application of the machinery of ΨDO leads astray. As already mentioned in the main text, the symbol of the operator $f\star$ is given by

$$\sigma[f\star](x,p) = f(x - \tfrac{1}{2}\Theta p).$$

Hence, $f\star$ is an infinitely smoothing operator if f is a Schwartz test function. In other words, the noncommutative Klein–Gordon operator $\Box^{\theta}_{A}$ differs from the free operator $\Box_0$ by an infinitely smoothing operator,

$$\begin{aligned}\sigma[\Box^{\theta}_{A}](x,p) &= -p^2 - 2ep^{\mu}A_{\mu}(x - \tfrac{1}{2}\Theta p) + ie(\partial^{\mu}A_{\mu})(x - \tfrac{1}{2}\Theta p) - e^2(A^{\mu}\star A_{\mu})(x - \tfrac{1}{2}\Theta p) \\ &= \sigma[\Box_0](x,p) + \text{smoothing}.\end{aligned}$$

One might therefore expect that the dependence on the fields A of the resolvent R is in the part that is not seen by an asymptotic expansion in p and hence does not contribute to the divergent behavior of the trace. For ΨDOs on *noncompact* manifolds M this line of reasoning must be taken with caution, since there might be additional divergent terms from the x-integration in the trace integral. This is nicely illustrated by the above computation and the following example. Consider the function

$$f(x,p) = e^{-x^2 e^{-p^2} - (1/4)p^2},$$

where x and p are one-dimensional variables. Clearly

$$|\partial_p^{\alpha}\partial_x^{\beta} f(x,p)| \leq C_{K,\alpha,\beta} e^{-(1/4)p^2}, \quad x \in K \subset \mathbb{R} \text{ compact}, \; p \in \mathbb{R},$$

hence f defines an infinitely smoothing operator. On the other hand,

$$\int_{\mathbb{R}} \mathrm{d}x\, f(x,p) = \sqrt{\pi} e^{(1/4)p^2},$$

and the operator f does have a diverging trace. Note that in this example, it is the noncompactness that yields the surprise. We conclude that even in the commutative case, the correspondence between the logarithmically divergent part of the trace and the residue needs some additional justification.

In the above calculation, however, the $p-x$ mixing in the arguments of the fields A_μ—which originates from the noncommutativity of the Moyal plane—makes it impossible to distinguish between the asymptotic p-expansion and an (infinitely smoothing) remainder. There, additional arguments are imperative. Observe, however, that our lines of reasoning above can be taken over to the commutative case, thereby solving the raised objection.

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